Clifford Algebra and Spacetime Geometry

Sandro Vitenti

This document summarizes key aspects of the Clifford algebra $\mathrm{Cl}(1,3)$, which encodes the geometric structure of Minkowski spacetime. We outline its basis, multivector structure, products, and important identities. The geometric algebra approach provides an algebraic description of spacetime, encompassing Lorentz transformations and reference frames.

Clifford Algebra of Spacetime

The algebra $\mathrm{Cl}(1,3)$ is generated by a set of 1-vectors e^μ satisfying the Minkowski metric $\eta\_\mu\nu = \mathrm{diag}(-1,1,1,1)$. Its elements are constructed from wedge products of these generators:

$$\mathrm{Cl}(1,3) = \mathrm{span}\left\{1, \mathsf{e}^\mu, \mathsf{e}^\mu \wedge \mathsf{e}^\nu, \mathsf{e}^\mu \wedge \mathsf{e}^\nu \wedge \mathsf{e}^\rho, \mathsf{e}^0 \wedge \mathsf{e}^1 \wedge \mathsf{e}^2 \wedge \mathsf{e}^3\right\}$$

The algebra has 16 independent elements: one scalar, four vectors, six bivectors, four trivectors, and one pseudoscalar. These are classified by grade: a scalar is grade-0, a vector is grade-1, a bivector is grade-2, a trivector is grade-3, and the pseudoscalar is grade-4. The full algebra includes linear combinations of such elements, known collectively as multivectors.

Each k-vector represents a specific geometric object: for instance, e^μ is a 1-vector (associated with directions), $\mathsf{e}^\mu \wedge \mathsf{e}^\nu$ is a bivector (representing oriented planes), and so on. This graded structure mirrors the decomposition of the algebra and plays a central role in defining its products and geometric interpretation.

For grade-1 elements $\omega, u \in \mathrm{Cl}(1,3)$, the symmetric (inner) and antisymmetric (exterior) products are defined as:

$$\omega \cdot u = \frac{1}{2}(\omega u + u\omega), \qquad \omega \wedge u = \frac{1}{2}(\omega u - u\omega)$$

These operations extract the grade-0 and grade-2 parts of the Clifford product, respectively. The algebra is defined by the inner products of the basis elements:

$$\mathsf{e}^\mu \cdot \mathsf{e}^\nu = \eta^{\mu\nu}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, used to raise and lower indices, including elements of the algebra and components of (multi)vectors.

Given vectors written in component form as $\omega = \omega^\mu \mathsf{e}_\mu$ and $u = u^\nu \mathsf{e}_\nu$, the inner product becomes:

$$\omega \cdot u = \omega^\mu u^\nu \eta_{\mu\nu}$$

while the exterior product is:

$$\omega \wedge u = (\omega^\mu u^\nu - \omega^\nu u^\mu)\, \mathsf{e}_\mu \wedge \mathsf{e}_\nu$$

The exterior product is antisymmetric and defines a bivector, corresponding to an oriented plane segment spanned by ω and u.

The Clifford algebra encodes the geometric structure of spacetime, including Lorentz transformations and the behavior of reference frames. By defining the inner product as the symmetric part of the Clifford product, the Minkowski metric emerges directly from the algebraic relations among basis vectors. This allows calculations in special relativity to be carried out through purely algebraic manipulations.

The basis vectors satisfy:

$$(\mathsf{e}^0)^2 = \mathsf{e}^0 \cdot \mathsf{e}^0 = -1, \qquad (\mathsf{e}^i)^2 = \mathsf{e}^i \cdot \mathsf{e}^i = 1 \quad (i = 1, 2, 3)$$

which reflects the signature of the Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. For distinct indices, the generators anticommute:

$$\mathsf{e}^\mu \mathsf{e}^\nu = -\mathsf{e}^\nu \mathsf{e}^\mu \qquad (\mu \neq \nu)$$

These relations define the algebraic structure that underlies spacetime geometry in the context of special relativity.

Bivectors and Their Algebra

Bivectors are grade-2 elements constructed as wedge products of vectors. They span a 6-dimensional subspace of $\text{Cl}(1, 3)$, with basis elements of the form $\mathsf{e}^\mu \wedge \mathsf{e}^\nu$ for $\mu < \nu$. By construction, they are antisymmetric:

$$\mathsf{e}^\mu \wedge \mathsf{e}^\nu = -\mathsf{e}^\nu \wedge \mathsf{e}^\mu$$

The square of a bivector depends on the causal character of the plane it spans. Bivectors involving the timelike basis vector e^0 (e.g., $\mathsf{e}^0 \wedge \mathsf{e}^1$) are called timelike and satisfy:

$$(\mathsf{e}^0 \wedge \mathsf{e}^1)^2 = 1$$

In contrast, bivectors composed of spatial vectors only (e.g., $\mathsf{e}^1 \wedge \mathsf{e}^2$) are spacelike and satisfy:

$$(\mathsf{e}^1 \wedge \mathsf{e}^2)^2 = -1$$

When manipulating bivectors, it is useful to note that for orthogonal vectors e^μ and e^ν, the wedge and Clifford products coincide:

$$\mathsf{e}^\mu \wedge \mathsf{e}^\nu = \mathsf{e}^\mu \mathsf{e}^\nu \quad (\mu \neq \nu).$$

For example,

$$(\mathsf{e}^0 \wedge \mathsf{e}^1)^2 = (\mathsf{e}^0 \mathsf{e}^1)^2 = \mathsf{e}^0 \mathsf{e}^1 \mathsf{e}^0 \mathsf{e}^1.$$

Using anticommutation, $\mathsf{e}^1 \mathsf{e}^0 = -\mathsf{e}^0 \mathsf{e}^1$, we have:

$$\mathsf{e}^0 \mathsf{e}^1 \mathsf{e}^0 \mathsf{e}^1 = -\mathsf{e}^0 \mathsf{e}^0 \mathsf{e}^1 \mathsf{e}^1 = -(\mathsf{e}^0)^2 (\mathsf{e}^1)^2.$$

Since $(\mathsf{e}^0)^2 = -1$ and $(\mathsf{e}^1)^2 = 1$, it follows that:

$$(\mathsf{e}^0 \wedge \mathsf{e}^1)^2 = 1.$$

This illustrates the general procedure for computing with multivectors: expand the product, apply anticommutation relations, and use the metric to evaluate contractions. This approach extends naturally to more complex expressions involving elements of arbitrary grade.

This distinction reflects the signature of the Minkowski metric and divides the six basis bivectors into two classes: three with positive square (timelike) and three with negative square (spacelike). Bivectors play a central role in representing infinitesimal Lorentz transformations, with spacelike bivectors generating spatial rotations and timelike bivectors generating boosts.

Trivectors

Trivectors are grade-3 elements formed by wedge products of three distinct vectors. They span a 4-dimensional subspace of $\text{Cl}(1, 3)$ and are fully antisymmetric. A basis consists of one purely spatial trivector,

$$\mathsf{e}^1 \wedge \mathsf{e}^2 \wedge \mathsf{e}^3,$$

and three mixed (temporal) trivectors,

$$\mathsf{e}^0 \wedge \mathsf{e}^1 \wedge \mathsf{e}^2, \qquad \mathsf{e}^0 \wedge \mathsf{e}^2 \wedge \mathsf{e}^3, \qquad \mathsf{e}^0 \wedge \mathsf{e}^1 \wedge \mathsf{e}^3.$$

Their squared norms reflect the causal nature of the planes they span. The purely spatial trivector satisfies:

$$(\mathsf{e}^1 \wedge \mathsf{e}^2 \wedge \mathsf{e}^3)^2 = -1,$$

while temporal trivectors satisfy:

$$(\mathsf{e}^0 \wedge \mathsf{e}^1 \wedge \mathsf{e}^2)^2 = 1.$$

Trivectors arise naturally in Clifford products involving three vectors, such as $\mathsf{e}^0\mathsf{e}^1\mathsf{e}^2$, and require careful attention to sign conventions due to their antisymmetry, e.g.,

$$\mathsf{e}^0\mathsf{e}^1\mathsf{e}^2 = -\mathsf{e}^1\mathsf{e}^0\mathsf{e}^2.$$

Although trivectors have fewer direct physical interpretations than bivectors, they are useful in defining duals and in describing oriented volumes in spacetime.

Pseudoscalar

The pseudoscalar is the unique grade-4 element of $\mathrm{Cl}(1,3)$:

$$I = \mathsf{e}^0 \wedge \mathsf{e}^1 \wedge \mathsf{e}^2 \wedge \mathsf{e}^3 = \mathsf{e}^0\mathsf{e}^1\mathsf{e}^2\mathsf{e}^3, \qquad I^2 = 1.$$

It represents the oriented 4-volume element in spacetime—not to be confused with the usual 3-dimensional spatial volume, and plays a central role in the geometric structure of the algebra.

The pseudoscalar anticommutes with all 1-vectors:

$$I\mathsf{e}^\mu = -\mathsf{e}^\mu I,$$

and acts as the generator of Hodge duality via multiplication. For example: - $I\mathsf{e}^\mu$ is a trivector, - $I(\mathsf{e}^\mu \wedge \mathsf{e}^\nu)$ yields a bivector, - $I \cdot 1 = I$ maps scalars to pseudoscalars.

As the oriented volume element, I performs the same operation as the Hodge star map, which is traditionally defined on components via contraction with the Levi-Civita symbol. For instance,

$$\mathsf{e}^1 I = \mathsf{e}^1(\mathsf{e}^0\mathsf{e}^1\mathsf{e}^2\mathsf{e}^3) = -\mathsf{e}^0\mathsf{e}^2\mathsf{e}^3$$

maps the vector e^1 to its dual trivector. This mirrors the way a cross product maps two vectors to a third in 3D, or more generally, how a k-vector is mapped to an $(n-k)$-vector in n-dimensional Hodge duality.

In $\mathrm{Cl}(1,3)$, I implements the following dualities: - Scalars $\leftrightarrow$ pseudoscalars, - Vectors $\leftrightarrow$ trivectors, - Bivectors $\rightarrow$ bivectors (interchanging timelike and spacelike types).

This operation will be shown explicitly in later sections and is especially useful in formulating Lorentz-covariant quantities such as the electromagnetic field.

Additional Notes

The Clifford algebra $\mathrm{Cl}(1,3)$ consists of 16 basis elements: 1 scalar, 4 vectors, 6 bivectors, 4 trivectors, and 1 pseudoscalar. This structure is isomorphic to the 4×4 matrix algebra used in Dirac's formulation of quantum mechanics, providing a natural framework for relativistic physics. Scalars are Lorentz-invariant, vectors transform as four-vectors under Lorentz transformations, and bivectors correspond to antisymmetric rank-2 tensors, such as the electromagnetic field tensor $F^{\mu\nu}$. The pseudoscalar $I = \mathsf{e}^0 \wedge \mathsf{e}^1 \wedge \mathsf{e}^2 \wedge \mathsf{e}^3$ represents the oriented 4-volume element of Minkowski spacetime and remains invariant under proper Lorentz transformations (determinant 1).

For computations in $\mathrm{Cl}(1,3)$, the following algebraic rules are essential:

1. The squares of basis vectors reflect the Minkowski metric: $(\mathsf{e}^0)^2 = -1$, $(\mathsf{e}^i)^2 = 1$ for spatial indices $i = 1, 2, 3$.
2. Basis vectors anticommute for distinct indices: $\mathsf{e}^\mu \mathsf{e}^\nu = -\mathsf{e}^\nu \mathsf{e}^\mu$ when $\mu \neq \nu$.
3. Permutations of generators in multivector products introduce sign changes due to antisymmetry, e.g., $\mathsf{e}^0\mathsf{e}^1\mathsf{e}^2\mathsf{e}^3 = -\mathsf{e}^1\mathsf{e}^0\mathsf{e}^2\mathsf{e}^3$.

These properties enable the algebraic derivation of key results in special relativity, such as Lorentz transformations, invariant intervals, and spacetime rotations, by manipulating multivectors within the $\mathrm{Cl}(1,3)$ framework. This approach unifies geometric and algebraic operations, simplifying calculations in relativistic physics.

Duality Relations

The pseudoscalar

$$I = \mathsf{e}^0 \wedge \mathsf{e}^1 \wedge \mathsf{e}^2 \wedge \mathsf{e}^3 = \mathsf{e}^0\mathsf{e}^1\mathsf{e}^2\mathsf{e}^3$$

implements the Hodge duality operation in $\mathrm{Cl}(1,3)$, mapping k-vectors to $(4-k)$-vectors. In particular, multiplication by I maps bivectors to their dual bivectors. For the basis bivectors, the duality relations are:

$$\begin{aligned} I(\mathsf{e}^0\mathsf{e}^1) &= \mathsf{e}^2\mathsf{e}^3, \\ I(\mathsf{e}^0\mathsf{e}^2) &= -\mathsf{e}^1\mathsf{e}^3, \\ I(\mathsf{e}^0\mathsf{e}^3) &= \mathsf{e}^1\mathsf{e}^2, \\ I(\mathsf{e}^1\mathsf{e}^2) &= -\mathsf{e}^0\mathsf{e}^3, \\ I(\mathsf{e}^1\mathsf{e}^3) &= \mathsf{e}^0\mathsf{e}^2, \\ I(\mathsf{e}^2\mathsf{e}^3) &= -\mathsf{e}^0\mathsf{e}^1. \end{aligned}$$

These follow from anticommutation relations and the property $I\mathsf{e}^\mu = -\mathsf{e}^\mu I$. For example,

$$I(\mathsf{e}^0\mathsf{e}^1) = (\mathsf{e}^0\mathsf{e}^1\mathsf{e}^2\mathsf{e}^3)(\mathsf{e}^0\mathsf{e}^1) = \mathsf{e}^2\mathsf{e}^3,$$

where anticommutation and metric contractions are used.

Duality in 3D

Within the 3D spatial subalgebra of $\mathrm{Cl}(1,3)$ generated by

$$\sigma^i \equiv \mathsf{e}^0\mathsf{e}^i, \quad i = 1, 2, 3,$$

the spatial pseudoscalar

$$I_3 = \sigma^1\sigma^2\sigma^3 = I$$

defines a 3D duality operation analogous to the cross product. This subalgebra uses timelike-spatial bivectors σ^i, which satisfy

$$\sigma^i\sigma^j = -\sigma^j\sigma^i, \quad (\sigma^i)^2 = 1,$$

and their symmetric product reproduces the standard Euclidean inner product:

$$\sigma^i \cdot \sigma^j \equiv \frac{\sigma^i\sigma^j + \sigma^j\sigma^i}{2} = \delta^{ij}.$$

The 3D cross product emerges naturally as

$$u \times v \equiv -I(u \wedge v),$$

for vectors $u, v \in \mathrm{span}\{\sigma^1, \sigma^2, \sigma^3\}$. For example,

$$\begin{aligned} \sigma^1 \times \sigma^2 &= -I(\sigma^1 \wedge \sigma^2) = \sigma^3, \\ \sigma^1 \times \sigma^3 &= -I(\sigma^1 \wedge \sigma^3) = -\sigma^2, \\ \sigma^2 \times \sigma^3 &= -I(\sigma^2 \wedge \sigma^3) = \sigma^1. \end{aligned}$$

Here, the wedge product $\sigma^i \wedge \sigma^j$ is a bivector, and multiplication by I maps it back to a vector, revealing the bivector nature of the cross product.

This framework provides a natural algebraic interpretation of the magnetic field in electromagnetism as a bivector transforming under spatial rotations consistent with the duality I_3, unifying the cross product with spacetime geometry.

Contraction with the Pseudoscalar

Contraction with the pseudoscalar I in $\mathrm{Cl}(1,3)$ implements the Hodge dual operation algebraically. For a k-vector A, the product AI produces a $(4-k)$-vector, preserving the graded structure of the algebra.

Specific examples include: - Scalars $c \in \mathbb{R}$ map to pseudoscalars via $cI = c\,\mathsf{e}^0\mathsf{e}^1\mathsf{e}^2\mathsf{e}^3$. - Vectors transform into trivectors, e.g.,

$$I\mathsf{e}^\mu = (\mathsf{e}^0\mathsf{e}^1\mathsf{e}^2\mathsf{e}^3)\mathsf{e}^\mu, \quad \text{with} \quad I\mathsf{e}^1 = \mathsf{e}^0\mathsf{e}^2\mathsf{e}^3.$$

- Bivectors map to bivectors with timelike and spacelike components interchanged, as discussed previously. - Trivectors map to vectors:

$$I(\mathsf{e}^\mu\mathsf{e}^\nu\mathsf{e}^\rho) = \pm\mathsf{e}^\sigma,$$

where e^σ is the complementary basis vector completing the 4-vector basis.

This multiplication by I corresponds to the Hodge star operator on differential forms, but here it is realized entirely through algebraic multiplication, eliminating the need for component-wise contractions with the Levi-Civita tensor.

For example, the electromagnetic field tensor

$$F = \frac{1}{2}F_{\mu\nu}\mathsf{e}^\mu\mathsf{e}^\nu$$

is mapped to its dual

$$\star F = IF,$$

which exchanges electric and magnetic components. This duality plays a central role in expressing Maxwell's equations covariantly in spacetime.

The algebraic simplicity of contraction with I exemplifies the unifying power of $\mathrm{Cl}(1,3)$ in linking geometric and physical structures.

Contraction with the Pseudoscalar

For a vector $v = v^\mu \mathsf{e}_\mu$, contraction with the pseudoscalar I produces its dual trivector:

$$Iv = I(v^\mu\mathsf{e}_\mu) = -\frac{v^\mu}{3!}\epsilon_{\mu\nu\rho\sigma}\,\mathsf{e}^\nu\mathsf{e}^\rho\mathsf{e}^\sigma,$$

where $\epsilon_{0123} = 1$ is the totally antisymmetric Levi-Civita symbol. This expresses the Hodge dual of v as a trivector in $\mathrm{Cl}(1,3)$.

Vector Notation and Tensor Relations

Geometric objects in $\mathrm{Cl}(1,3)$ can be represented with tensor components as follows: - Trivector:

$$T = \frac{1}{3!}T^{\mu\nu\alpha}\mathsf{e}_\mu \wedge \mathsf{e}_\nu \wedge \mathsf{e}_\alpha,$$

- Bivector:

$$B = \frac{1}{2!}B^{\mu\nu}\mathsf{e}_\mu \wedge \mathsf{e}_\nu,$$

- Pseudoscalar:

$$\alpha I, \quad \alpha \in \mathbb{R}.$$

The geometric product of vectors $v = v^\mu \mathsf{e}_\mu$ and $w = w^\nu \mathsf{e}_\nu$ decomposes into scalar (inner) and bivector (exterior) parts:

$$vw = v^\mu w^\nu \left(\mathsf{e}_\mu \cdot \mathsf{e}_\nu + \mathsf{e}_\mu \wedge \mathsf{e}_\nu\right).$$

The Faraday tensor F, encoding the electromagnetic field, is naturally a bivector in $\mathrm{Cl}(1,3)$:

$$F = \frac{1}{2} F_{\mu\nu} \mathsf{e}^\mu \wedge \mathsf{e}^\nu.$$

Its components combine electric fields (associated with bivectors $\mathsf{e}^0 \wedge \mathsf{e}^i$) and magnetic fields (associated with spatial bivectors $\mathsf{e}^i \wedge \mathsf{e}^j$), totaling six independent components. Under Lorentz transformations, electric and magnetic parts mix, but F as a bivector remains invariant, unifying electric and magnetic fields within the algebraic framework of spacetime geometry.

Algebra of Bivectors

If two bivectors ω and u satisfy $\omega \wedge u = 0$, they commute under the Clifford product:

$$\omega u = u\omega.$$

Conversely, if they are orthogonal in the sense that their inner product vanishes, $\omega \cdot u = 0$, they anticommute:

$$\omega u = -u\omega.$$

In this case, their product squares as

$$(\omega u)^2 = -\omega^2 u^2,$$

with the sign depending on the signatures of ω^2 and u^2: - If $\omega^2 < 0$ and $u^2 > 0$, then $-\omega^2 u^2 > 0$. - If both $\omega^2 < 0$ and $u^2 < 0$, then $-\omega^2 u^2 < 0$.

The exponential of a bivector generates Lorentz transformations:

$$e^{\omega \wedge u} = \cosh\left(\sqrt{-\omega^2 u^2}\right) + \frac{\omega \wedge u}{\sqrt{-\omega^2 u^2}} \sinh\left(\sqrt{-\omega^2 u^2}\right).$$

Bivectors generate boosts or rotations depending on their causal character. For instance, the timelike bivector $\mathsf{e}^0 \wedge \mathsf{e}^1$, with $(\mathsf{e}^0 \wedge \mathsf{e}^1)^2 = 1$, generates boosts:

$$\mathsf{e}^{\lambda(\mathsf{e}^0 \wedge \mathsf{e}^1)} = \cosh\lambda + (\mathsf{e}^0 \wedge \mathsf{e}^1)\sinh\lambda.$$

Applying this boost to the vector $w = \mathsf{e}^0$ gives

$$\mathsf{e}^{-\lambda(\mathsf{e}^0 \wedge \mathsf{e}^1)} \mathsf{e}^0 \mathsf{e}^{\lambda(\mathsf{e}^0 \wedge \mathsf{e}^1)} = (\cosh\lambda)\mathsf{e}^0 + (\sinh\lambda)\mathsf{e}^1,$$

corresponding to a Lorentz boost along e^1 with $\cosh\lambda = \gamma$, $\sinh\lambda = \gamma\beta$.

In contrast, a spacelike bivector such as $\mathsf{e}^1 \wedge \mathsf{e}^2$, with $(\mathsf{e}^1 \wedge \mathsf{e}^2)^2 = -1$, generates spatial rotations:

$$\mathsf{e}^{\lambda(\mathsf{e}^1 \wedge \mathsf{e}^2)} = \cos\lambda + (\mathsf{e}^1 \wedge \mathsf{e}^2)\sin\lambda.$$

Electromagnetism in Clifford Algebra

The electromagnetic field is naturally represented by the Faraday bivector

$$F = \frac{1}{2} F_{\mu\nu} \mathsf{e}^\mu \wedge \mathsf{e}^\nu,$$

which contains six components: three electric (associated with bivectors $\mathsf{e}^0 \wedge \mathsf{e}^i$) and three magnetic (associated with $\mathsf{e}^i \wedge \mathsf{e}^j$). Under Lorentz transformations, electric and magnetic parts mix, but the bivector F itself remains invariant. This formalism unifies electric and magnetic fields as frame-dependent projections of the electromagnetic field tensor, simplifying the formulation of electromagnetism and related calculations such as the Lorentz force. Further applications will be developed in the course.

Exercises

To improve your understanding of Cl(1, 3), work through these exercises:

1. Squares of Basis Bivectors
   Compute the square of the following bivectors and classify each as timelike, spacelike, or lightlike:
   a) $(\mathsf{e}^0 \wedge \mathsf{e}^1)^2$
   b) $(\mathsf{e}^1 \wedge \mathsf{e}^2)^2$
   c) $(\mathsf{e}^2 \wedge \mathsf{e}^3)^2$
   d) $(\mathsf{e}^0 \wedge \mathsf{e}^2)^2$
2. Duals with the Pseudoscalar
   Calculate the duals of bivectors by multiplying with the pseudoscalar I:
   a) $I(\mathsf{e}^0\mathsf{e}^1)$
   b) $I(\mathsf{e}^1\mathsf{e}^2)$
   c) $I(\mathsf{e}^0\mathsf{e}^2)$
   d) $I(\mathsf{e}^2\mathsf{e}^3)$
   Verify that these results are consistent with the duality relations in Cl(1, 3).
3. Squares of Trivectors
   Compute the squares of these trivectors and determine their causal character:
   a) $(\mathsf{e}^0 \wedge \mathsf{e}^1 \wedge \mathsf{e}^2)^2$
   b) $(\mathsf{e}^1 \wedge \mathsf{e}^2 \wedge \mathsf{e}^3)^2$
4. Pseudoscalar Commutation
   Verify the anticommutation relation with the pseudoscalar for each basis vector:

$$I\mathsf{e}^\mu = -\mathsf{e}^\mu I, \quad \mu = 0, 1, 2, 3.$$

   Identify the grade and type of the multivector $I\mathsf{e}^\mu$.
5. Exponential of a Bivector
   For $\omega = \mathsf{e}^0$, $u = \mathsf{e}^1$:
   a) Compute the wedge product $\omega \wedge u$.
   b) Calculate $(\omega \wedge u)^2$.
   c) Write down the exponential $\mathsf{e}^{\omega \wedge u}$ and interpret whether it corresponds to a boost or a rotation.
6. Anticommutators and Commutators
   Calculate and interpret:
   a) The anticommutator $\mathsf{e}^0\mathsf{e}^1 + \mathsf{e}^1\mathsf{e}^0$.
   b) The commutator $\mathsf{e}^1\mathsf{e}^2 - \mathsf{e}^2\mathsf{e}^1$.
   Relate these to the inner (symmetric) and exterior (antisymmetric) products of vectors.

Inversion and Hermitian Conjugates

In Cl(1, 3), the reversion (also called transpose) is an involutive operation defined on multivectors by:

$$(ab)^{\mathrm{T}} = b^{\mathrm{T}} a^{\mathrm{T}},$$

for any multivectors a and b. Reversion reverses the order of the product but leaves individual vectors and scalars unchanged:

$$(\mathsf{e}^\mu)^{\mathrm{T}} = \mathsf{e}^\mu, \quad (\alpha)^{\mathrm{T}} = \alpha, \quad \text{for any } \alpha \in \mathbb{R}.$$

For the pseudoscalar $I = \mathsf{e}^0\mathsf{e}^1\mathsf{e}^2\mathsf{e}^3$:

$$I^{\mathrm{T}} = (\mathsf{e}^0\mathsf{e}^1\mathsf{e}^2\mathsf{e}^3)^{\mathrm{T}} = \mathsf{e}^3\mathsf{e}^2\mathsf{e}^1\mathsf{e}^0 = I,$$

since reversing all four basis vectors introduces an even number of sign changes.

For a bivector $B = \frac{1}{2}B_{\mu\nu}\mathsf{e}^\mu \wedge \mathsf{e}^\nu$, the reversion changes the sign:

$$B^{\mathrm{T}} = \frac{1}{2}B_{\mu\nu}(\mathsf{e}^\nu \wedge \mathsf{e}^\mu) = -\frac{1}{2}B_{\mu\nu}\mathsf{e}^\mu \wedge \mathsf{e}^\nu = -B,$$

reflecting the antisymmetry of bivectors under reversion.

Exponentials of Bivectors

The exponential of a bivector B is defined through the usual power series:

$$e^{\alpha B} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} B^n,$$

with $\alpha \in \mathbb{R}$. The closed form depends on the square of B, and can be derived by separating the even and odd terms:

- For $B^2 = +1$, (e.g., $B = \mathsf{e}^0 \wedge \mathsf{e}^1$) the exponential is given by:

$$\begin{aligned} e^{\alpha B} &= \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} B^n \\ &= \sum_{n=0}^{\infty} \left(\frac{\alpha^{2n}}{(2n)!} B^{2n} + \frac{\alpha^{2n+1}}{(2n+1)!} B^{2n+1} \right) \\ &= \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n)!} + B \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n+1)!} \\ &= \cosh(\alpha) + B \sinh(\alpha). \end{aligned}$$

- For $B^2 = -1$, (e.g., $B = \mathsf{e}^2 \wedge \mathsf{e}^3$) the exponential is given by:

$$\begin{aligned} e^{\alpha B} &= \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} B^n \\ &= \sum_{n=0}^{\infty} \left(\frac{\alpha^{2n}}{(2n)!} B^{2n} + \frac{\alpha^{2n+1}}{(2n+1)!} B^{2n+1} \right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n}}{(2n)!} + B \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1}}{(2n+1)!} \\ &= \cos(\alpha) + B \sin(\alpha). \end{aligned}$$

Lorentz Transformations with Rotors

Lorentz transformations are implemented using rotors:

$$R = e^{\alpha B/2},$$

where B is a bivector (e.g., $\mathsf{e}^0 \wedge \mathsf{e}^1$ for boosts or $\mathsf{e}^1 \wedge \mathsf{e}^2$ for rotations), and $\alpha \in \mathbb{R}$ is a transformation parameter. The rotor acts on a vector u via the sandwich product:

$$u \mapsto RuR^{\mathrm{T}}, \quad \text{with} \quad R^{\mathrm{T}} = e^{-\alpha B/2}.$$

This ensures that the result is a vector, since:

$$(RuR^{\mathrm{T}})^{\mathrm{T}} = RuR^{\mathrm{T}}.$$

In contrast, a simple left multiplication like Ru is not a vector:

$$(Ru)^{\mathrm{T}} = uR^{\mathrm{T}} \neq Ru.$$

The use of both R and R^{T} in the transformation law necessitates the factor $\alpha/2$ in the exponent, ensuring the correct group composition law:

$$R(\alpha_1)R(\alpha_2) = R(\alpha_1 + \alpha_2).$$

Spinors and Lorentz Rotations

Rotors exhibit spinorial behavior. For example, a 2π rotation gives:

$$R(2\pi) = \cos(\pi) + B\sin(\pi) = -1 = -R(0),$$

indicating that a full 4π rotation is needed to return to the identity:

$$R(4\pi) = 1.$$

This is the hallmark of spin-1/2 behavior. Spinors arise naturally in the even subalgebra of $\mathrm{Cl}(1,3)$, forming the group $\mathrm{Spin}(1,3)$, which is a double cover of the Lorentz group $\mathrm{SO}(1,3)$. The same rotor R that acts on vectors as:

$$u \mapsto RuR^{\mathrm{T}},$$

acts on spinors ψ (e.g., four-component complex vectors) as:

$$\psi \mapsto R\psi.$$

Note, however, that this approach would require introducing a matrix representation of the algebra. While this is a common route, it is not strictly necessary. Instead, we can represent the "vector" ψ directly as an element of $\mathrm{Cl}(1,3)$, more precisely, as an even multivector.

The pseudoscalar $I = \mathsf{e}_0\mathsf{e}_1\mathsf{e}_2\mathsf{e}_3$ anticommutes with all vectors and satisfies $I^2 = -1$, playing a role analogous to the complex unit i in the Dirac algebra. In the spinor representation, it appears in constructions involving γ^5 and chirality.

Reference Frames and 4-Velocity

An observer's 4-velocity is $w^\mu = (1, 0, 0, 0)$, or $w = \mathsf{e}_0$, with $w^2 = -1$. A particle's 4-velocity is:

$$u^\mu = \left(\gamma, \gamma\frac{v^1}{c}, \gamma\frac{v^2}{c}, \gamma\frac{v^3}{c}\right), \quad \gamma = (1 - v^2/c^2)^{-1/2},$$

so $u = u^\mu \mathsf{e}_\mu$, with:

$$u^2 = -\gamma^2(1 - v^2/c^2) = -1, \quad u \cdot w = -\gamma.$$

Trajectories and Decomposition

A spacetime trajectory is:

$$x = x^\mu \mathsf{e}_\mu = x^0 \mathsf{e}_0 + x^i \mathsf{e}_i, \quad x \cdot w = -x^0.$$

The spatial projection orthogonal to $w = \mathsf{e}_0$:

$$x_\perp = \frac{x + wxw}{2} = x^i \mathsf{e}_i,$$

since:

$$wxw = -x^0 \mathsf{e}_0 + x^i \mathsf{e}_i, \quad x + wxw = 2x^i \mathsf{e}_i.$$

Exercises

Basic Comprehension

1. Show that for $u = u^\mu \mathsf{e}_\mu$, we have $u^2 = -1$ and $u \cdot w = -\gamma$ for $w = \mathsf{e}_0$.
2. Compute the spatial projection $x_\perp = \frac{x+wxw}{2}$ for $x = x^0 \mathsf{e}_0 + x^1 \mathsf{e}_1$, and check that $x_\perp \cdot w = 0$.
3. Let $B = \mathsf{e}^2 \wedge \mathsf{e}^3$. Compute B^2 and verify the closed form of $e^{\theta B}$.
4. Prove that for any bivector B and scalar α, we have:

$$\left(e^{\alpha B}\right)^{\mathrm{T}} = e^{-\alpha B}.$$

5. Let $R = e^{\theta B/2}$ for a bivector B with $B^2 = -1$. Show that $R^{\mathrm{T}} = R^{-1}$.

Applications

6. Show that the action $u \mapsto R u R^{\mathrm{T}}$ preserves the norm: $(R u R^{\mathrm{T}})^2 = u^2$.
7. Let $B = \mathsf{e}^0 \wedge \mathsf{e}^1$ and $R = e^{\eta B/2}$ with rapidity η. Compute $R \mathsf{e}^0 R^{\mathrm{T}}$ and interpret the result as a Lorentz boost along the x^1 direction.
8. Verify explicitly that a full 2π rotation gives $R = -1$ for $B = \mathsf{e}^1 \wedge \mathsf{e}^2$.
9. Let $x = x^0 \mathsf{e}_0 + x^1 \mathsf{e}_1$ and $w = \mathsf{e}_0$. Show that the time component of x is recovered by:

$$t = -x \cdot w.$$

10. Let $B = \mathsf{e}^1 \wedge \mathsf{e}^2$. Compute $R = \cos(\theta/2) + B \sin(\theta/2)$ and then compute R^2. What is the result? Why does this make sense?
11. Show that the pseudoscalar $I = \mathsf{e}_0 \mathsf{e}_1 \mathsf{e}_2 \mathsf{e}_3$ anticommutes with all vectors: $I \mathsf{e}^\mu = -\mathsf{e}^\mu I$, and deduce that it also anticommutes with all bivectors.
12. Let $x = x^\mu \mathsf{e}_\mu$ and $w = \mathsf{e}_0$. Show that $x = -tw + x_\perp$, where $t = -x \cdot w$ and $x_\perp = \frac{x+wxw}{2}$.