Green Functions

Sandro Vitenti

Green's Functions in One Dimension

Green's functions provide a powerful and general method for solving linear differential equations with prescribed boundary conditions. The core idea is to express the solution as a convolution between a Green's function and a source term. This method is particularly useful in both ordinary and partial differential equations, especially in physics and engineering contexts where fundamental solutions or propagators are central tools.

The exercises and derivations that follow develop the concept of Green's functions from basic examples in one dimension, building toward general properties such as symmetry, boundary behavior, and their role in solving inhomogeneous equations. These one-dimensional constructions are then extended to higher dimensions, where the Green's functions become essential in solving the Poisson and Laplace equations.

The step-by-step approach mirrors the structure used in previous sections, especially those on the Dirac delta distribution, highlighting how distributions and fundamental solutions are interconnected. This transition from delta sequences to Green's functions reveals the deeper structure behind linear differential operators and their inverses.

Exercises

1. The inverse of the derivative of a function $f(x)$ for $x \in (a,b)$, with $f(a) = 0 = f(b)$, can be written as:

$$f(x) = \int_a^x f'(y)\,\mathrm{d}y, \qquad f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(y)\,\mathrm{d}y.$$

   1. Show how to rewrite these expressions using the Heaviside step function $\theta(x)$:

$$\theta(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

      That is, $\theta(x)$ acts as the Green's function for the first derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^b \theta(x-y) f(y)\,\mathrm{d}y = f(x).$$

   2. Compute

$$G(x-y) \equiv \int_a^x \theta(z-y)\,\mathrm{d}z, \qquad y \in (a,b).$$

      and show that $G(x)$ is the Green's function for the second derivative:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_a^b G(x-y) f(y)\,\mathrm{d}y = f(x).$$

   3. Prove the inverse relation:

$$f(x) = \int_a^b G(x-y)\frac{\mathrm{d}^2}{\mathrm{d}y^2} f(y)\,\mathrm{d}y.$$

4. Express $G(x, y)$ in terms of $\sigma(x, y) = |x - y|$.

2. One-dimensional Green's functions. Derive Green's identities for the operator $\hat{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on the interval (a, b), assuming $\phi(a) = \phi(b) = 0$.

   1. Start by verifying the product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\phi\frac{\mathrm{d}\psi}{\mathrm{d}x}\right) = \frac{\mathrm{d}\phi}{\mathrm{d}x}\frac{\mathrm{d}\psi}{\mathrm{d}x} + \phi\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}.$$

   2. Integrate both sides over (a, b) and use the boundary condition $\phi(a) = \phi(b) = 0$ to show:

$$\int_a^b \phi(x)\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}(x)\,\mathrm{d}x = -\int_a^b \frac{\mathrm{d}\phi}{\mathrm{d}x}(x)\frac{\mathrm{d}\psi}{\mathrm{d}x}(x)\,\mathrm{d}x.$$

   This is Green's first identity.

   3. Swap ϕ and ψ, perform the same computation, and subtract the two identities to obtain Green's second identity:

$$\int_a^b \left[\phi(x)\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}(x) - \psi(x)\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}(x)\right]\,\mathrm{d}x = 0.$$

   4. Now assume $\psi(x) = G(x, y)$ is the Green's function for the operator, and let $\phi(x)$ be a given function. Use the identity above to obtain the formal solution of the equation

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x) = s(x)$$

   in terms of the Green's function:

$$f(x) = \int_a^b G(x, y)s(y)\,\mathrm{d}y.$$

   5. Using the previous result, show that the Green's function is:

$$G(x - y) = \frac{|x - y|}{2} + \alpha(x - y) + \beta.$$

   6. Determine α and β by imposing the boundary conditions $G(x - y) = 0$ at $x = a$ and $x = b$ for fixed $y \in (a, b)$.

   7. Use the Green's function to solve $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x) = 1$ on (a, b) and show that

$$f(x) = -\frac{1}{2}(x - a)(x - b).$$

Green's Functions in Multiple Dimensions

The concepts developed in one dimension generalize naturally to higher dimensions, where Green's functions play a central role in solving elliptic partial differential equations such as the Poisson and Laplace equations. In multiple dimensions, the Green's function depends on the spatial distance between points and reflects the geometry of the underlying space.

This section explores how the Laplacian operator behaves under these generalizations, how to regularize singular Green's functions, and how Green's identities lead to integral representations of solutions. These results form the basis for classical potential theory and have applications in fields ranging from electrostatics to quantum mechanics.

Exercises

1. The Green's function for the second derivative in one dimension is proportional to the distance $\sigma(x,y) = |x-y|$. In three dimensions, define:

$$\sigma(\mathbf{x},\mathbf{y}) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2},$$

   where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$.

   1. Show that the Laplacian in $\mathbb{R}^3$,

$${}^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2},$$

      satisfies:

$${}^2 \left(\frac{1}{\sigma(\mathbf{x},\mathbf{y})} \right) = 0 \quad \text{for } \mathbf{x} \neq \mathbf{y}.$$

   2. Regularize the Green's function by introducing $\varepsilon > 0$:

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = -\frac{1}{4\pi} \frac{1}{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2+\varepsilon^2}}.$$

      Show that:

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} {}^2 G_\varepsilon(\mathbf{x},\mathbf{y})\, \mathrm{d}^3\mathbf{x} = 1.$$

2. Derive Green's identities for the Laplacian.

   1. Derive the following Green's identities:

$$\oint_{\partial\Omega} \phi\, \psi \cdot \mathrm{d}\mathbf{S} = \int_{\Omega} \left(\phi \cdot \psi + \phi\, {}^2\psi \right) \mathrm{d}^3\mathbf{x},$$

$$\oint_{\partial\Omega} \left(\phi\, \psi - \psi\, \phi \right) \cdot \mathrm{d}\mathbf{S} = \int_{\Omega} \left(\phi\, {}^2\psi - \psi\, {}^2\phi \right) \mathrm{d}^3\mathbf{x},$$

      where Ω is a volume bounded by the surface $\partial\Omega$.

   2. Show how to use the second Green's identity together with the Green's function for the Laplacian to obtain the solution to the Poisson equation.