Dirac Delta

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The Dirac delta distribution is a central object in mathematical physics. Although not a function in the classical sense, it can be rigorously defined as the limit of a sequence of well-behaved functions. One common construction uses the normal (Gaussian) distribution with vanishing variance. In this section, we explore this limiting process and use it to derive properties of the delta distribution.

Exercises

1. Consider the normal distribution:
$$p_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/(2\sigma^2)},$$
where $\sigma > 0$ is the standard deviation.
    1. Show that $p_\sigma(x)$ is normalized:
    $$\int_{-\infty}^{\infty} p_\sigma(x)\,\mathrm{d}x = 1.$$
    2. Compute the expected values of odd powers of x:
    $$\langle x^{2n+1} \rangle = \int_{-\infty}^{\infty} x^{2n+1} p_\sigma(x)\,\mathrm{d}x.$$
    3. Compute the expected values of even powers of x:
    $$\langle x^{2n} \rangle = \int_{-\infty}^{\infty} x^{2n} p_\sigma(x)\,\mathrm{d}x.$$
    4. Use the results above to show:
    $$\int_{-\infty}^{\infty} f(x) p_\sigma(x)\,\mathrm{d}x = f(0) + \frac{\sigma^2}{2} f''(0) + \frac{\sigma^4}{8} f^{(4)}(0) + \cdots,$$
    where derivatives are with respect to x. Conclude that the limit $\sigma \to 0$ yields the Dirac delta:
    $$\lim_{\sigma \to 0} p_\sigma(x) = \delta(x).$$