Thomson Scattering

Sandro Vitenti

These notes use the conventions fixed earlier in the course, and build directly on the covariant radiation field derived in Radiation from a Moving Charge and the energy flux introduced in Energy of the Electromagnetic Field. The goal is to go from the covariant field of an accelerated charge, through its non-relativistic (local rest-frame) limit, to the classical differential and total cross sections for scattering of light by a free charge.

1 Setup

A free charge q of mass m (an electron, in practice) sits at rest at the origin and is illuminated by a monochromatic plane wave,

$$\mathbf{E}_{\text{inc}}(t) = E_0\,\hat{\epsilon}\,\cos(\omega t), \qquad \mathbf{B}_{\text{inc}}(t) = \frac{1}{c}\,\hat{\mathbf{k}} \times \mathbf{E}_{\text{inc}}(t),$$

with $\hat{\epsilon} \perp \hat{\mathbf{k}}$, evaluated at the charge's location. Two approximations define the classical (Thomson) regime:

- **non-relativistic motion**: the charge's response velocity $v \ll c$, so terms of order v/c are dropped;
- **no radiation reaction**: the energy radiated per cycle is a small fraction of the charge's kinetic energy, so the incident field alone drives the motion.

2 Equation of motion

The Lorentz force on the charge is $q(\mathbf{E}_{\text{inc}} + \mathbf{v} \times \mathbf{B}_{\text{inc}})$. Since $|\mathbf{B}_{\text{inc}}| = |\mathbf{E}_{\text{inc}}|/c$, the magnetic term is smaller than the electric term by a factor $v/c \ll 1$ and is dropped, leaving

$$m\dot{\mathbf{v}} = q\mathbf{E}_{\text{inc}}(t), \qquad \dot{\mathbf{v}}(t) = \frac{q}{m} E_0\,\hat{\epsilon}\,\cos(\omega t).$$

The charge oscillates along the fixed direction $\hat{\epsilon}$ with acceleration amplitude $a_0 \equiv qE_0/m$; since $a_0/\omega \ll c$ in the non-relativistic regime, the charge's excursion is tiny compared to the wavelength, and it radiates essentially from a fixed point (the dipole approximation).

3 From the covariant field to the local frame

The radiation field of an accelerated charge, valid for any velocity, was written covariantly as

$$F^{\text{rad}}_{\mu\nu} = \frac{\mu_0 q}{4\pi (u\cdot\Delta x)^2}\left[\Delta x_\mu a_\nu - \Delta x_\nu a_\mu - \frac{\Delta x \cdot a}{u \cdot \Delta x}(\Delta x_\mu u_\nu - \Delta x_\nu u_\mu)\right]_{\tau=\tau_r},$$

with $a^\mu = \mathrm{d}u^\mu/\mathrm{d}\tau$. Here this is specialized to a charge that is instantaneously at rest, as is the case for the non-relativistic oscillation driven by the incident wave.

At the instant the charge is at rest, $u^\mu = (1, \mathbf{0})$ and, since $u_\mu a^\mu = 0$ forces $a^0 = 0$ there, $a^\mu = (0, \dot{\mathbf{v}}/c)$ exactly (this is the same identification used for the Larmor formula). With $\Delta x^\mu = (R, R\hat{\mathbf{n}})$, $R = |\mathbf{x}|$,

$$u \cdot \Delta x = -R, \qquad \Delta x \cdot a = R\,\frac{\hat{\mathbf{n}}\cdot\dot{\mathbf{v}}}{c}.$$

Substituting into $F_{0i}|_{\text{rad}} = -E_i/c$ and simplifying (the two $1/R$-carrying terms combine; the $\Delta x_\mu u_\nu$ term contributes $R\hat{n}_i$ using $u_i = 0, u_0 = -1$) gives

$$\boxed{\mathbf{E}_{\text{rad}}(\mathbf{x}, t) = \frac{\mu_0 q}{4\pi R}\Big[\dot{\mathbf{v}} - (\hat{\mathbf{n}}\cdot\dot{\mathbf{v}})\hat{\mathbf{n}}\Big]_{t_r} = -\frac{\mu_0 q}{4\pi R}\,\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\dot{\mathbf{v}})\Big|_{t_r},}$$

the familiar non-relativistic dipole radiation field, transverse to $\hat{\mathbf{n}}$ by construction. Writing χ for the angle between $\hat{\mathbf{n}}$ and $\dot{\mathbf{v}}$, its magnitude is

$$|\mathbf{E}_{\text{rad}}| = \frac{\mu_0 q\,|\dot{\mathbf{v}}|}{4\pi R}\sin\chi.$$

4 Radiated power and a consistency check

In the radiation zone, $u = \varepsilon_0 E^2$ and $\mathbf{S} = cu\,\hat{\mathbf{n}}$ (from the Radiation zone section), so the power radiated per solid angle is

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = R^2\,\mathbf{S}\cdot\hat{\mathbf{n}} = R^2 c\varepsilon_0|\mathbf{E}_{\text{rad}}|^2 = \frac{\mu_0 q^2\dot{v}^2}{16\pi^2 c}\sin^2\chi,$$

using $1/(4\pi\varepsilon_0) = \mu_0 c^2/(4\pi)$. As a check, integrating over solid angle with $\int\sin^2\chi\,\mathrm{d}\Omega = 8\pi/3$ recovers exactly the non-relativistic Larmor formula already derived,

$$P = \int\frac{\mathrm{d}P}{\mathrm{d}\Omega}\,\mathrm{d}\Omega = \frac{\mu_0 q^2\dot{v}^2}{16\pi^2 c}\cdot\frac{8\pi}{3} = \frac{\mu_0 q^2\dot{v}^2}{6\pi c},$$

matching the boxed result $P = \mu_0 q^2\dot{v}^2/(6\pi c)$ obtained independently from the covariant radiation tensor.

5 Differential cross section

The incident wave carries energy density $u_{\text{inc}} = \varepsilon_0 E_{\text{inc}}^2$ and flux $S_{\text{inc}} = cu_{\text{inc}}$ (same radiation-zone relations, since a plane wave is locally the same kind of transverse field); its time average over a cycle is

$$\langle S_{\text{inc}}\rangle = c\varepsilon_0\langle E_{\text{inc}}^2\rangle = \tfrac{1}{2}c\varepsilon_0 E_0^2.$$

The differential cross section is defined as scattered power per solid angle, divided by incident flux:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \equiv \frac{\langle\mathrm{d}P/\mathrm{d}\Omega\rangle}{\langle S_{\text{inc}}\rangle}.$$

Using $\dot{\mathbf{v}} = (q/m)\mathbf{E}_{\text{inc}}$ from the equation of motion, $\langle\dot{v}^2\rangle = (q/m)^2\langle E_{\text{inc}}^2\rangle = (q/m)^2E_0^2/2$, so

$$\left\langle\frac{\mathrm{d}P}{\mathrm{d}\Omega}\right\rangle = \frac{\mu_0 q^2}{16\pi^2 c}\cdot\frac{q^2E_0^2}{2m^2}\sin^2\chi,$$

and dividing by $\langle S_{\text{inc}}\rangle = c\varepsilon_0 E_0^2/2$, with $1/\varepsilon_0 = \mu_0 c^2$,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\mu_0 q^2}{4\pi m}\right)^2\sin^2\chi.$$

Defining the **classical radius** of the charge,

$$r_q \equiv \frac{\mu_0 q^2}{4\pi m} = \frac{q^2}{4\pi\varepsilon_0 mc^2},$$

this is

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = r_q^2\sin^2\chi,}$$

with χ the angle between the observation direction $\hat{\mathbf{n}}$ and the (fixed) oscillation direction $\hat{\epsilon}$: scattering is strongest broadside to the oscillation and vanishes along it, exactly like a driven dipole antenna.

6 Unpolarized incident light

For unpolarized incident light, average $\sin^2\chi$ over the two independent transverse polarizations. Take $\hat{\mathbf{k}} = \hat{\mathbf{z}}$ and, without loss of generality, the observation direction in the x-z plane at scattering angle θ from $\hat{\mathbf{k}}$, $\hat{\mathbf{n}} = \sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{z}}$. For $\hat{\epsilon} = \hat{\mathbf{x}}$, $\sin^2\chi = 1 - (\hat{\mathbf{n}}\cdot\hat{\mathbf{x}})^2 = \cos^2\theta$; for $\hat{\epsilon} = \hat{\mathbf{y}}$, $\hat{\mathbf{n}}\cdot\hat{\mathbf{y}} = 0$ so $\sin^2\chi = 1$. Averaging the two,

$$\langle \sin^2\chi \rangle = \frac{\cos^2\theta + 1}{2},$$

so that

$$\boxed{\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{\mathrm{unpol}} = \frac{r_q^2}{2}\left(1 + \cos^2\theta\right),}$$

the classical Thomson formula, with θ now the angle between the scattered and incident directions.

7 Total cross section

Integrating over solid angle, with $x = \cos\theta$,

$$\sigma_T = \frac{r_q^2}{2}\int_0^{2\pi}\mathrm{d}\phi\int_{-1}^{1}(1 + x^2)\,\mathrm{d}x = \pi r_q^2\left[x + \frac{x^3}{3}\right]_{-1}^{1} = \pi r_q^2\cdot\frac{8}{3},$$

$$\boxed{\sigma_T = \frac{8\pi}{3}r_q^2.}$$

For an electron ($q = e$, $m = m_e$), r_q is the classical electron radius and $\sigma_T \approx 6.65\times10^{-29}\,\mathrm{m}^2$ is the Thomson cross section.

8 Validity

This is a purely classical, elastic calculation: the scattered light has the same frequency ω as the incident light, with no recoil. It holds provided:

- $v \ll c$ throughout the motion (already used to drop the magnetic force and to evaluate a^μ at $u^\mu = (1, \mathbf{0})$);
- the photon energy $\hbar\omega$ is much smaller than mc^2, so that the quantum recoil corrections of Compton scattering are negligible;
- the charge is genuinely free (or driven far from any resonance), so that no restoring force competes with the incident field in the equation of motion.

Feel free to create issues, ask questions, or suggest improvements in the GitHub repository.