Radiation from a Moving Charge

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These notes closely follow the original lecture derivation, with only minor additions for clarity, and use the conventions fixed earlier in the course.

1 Maxwell equations in covariant form

Starting from Maxwell's equations,

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu,$$

with $F^{0i} = E^i/c$, $F^{ij} = \epsilon^{ijk} B_k$, and $J^\mu = (c\rho, \mathbf{J})$ as fixed in the conventions, one recovers the usual Maxwell equations component by component exactly as in Maxwell Equations in Covariant Form.

Introducing the four-potential,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

the $0i$ component reads $F_{0i} = \partial_0 A_i - \partial_i A_0 = \frac{1}{c}\partial_t A_i - \partial_i A_0$; comparing with $F_{0i} = -E_i/c$ (the lowered version of $F^{0i} = E^i/c$) gives

$$\frac{E_i}{c} = \partial_i A_0 - \frac{1}{c}\frac{\partial A_i}{\partial t},$$

and, from the spatial components,

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

In the electrostatic limit,

$$\frac{\partial A_i}{\partial t} = 0, \qquad E_i = \partial_i (cA_0) = -\partial_i \phi,$$

using $cA_0 = -\phi$ from $A^\mu = (\phi/c, \mathbf{A})$.

2 Wave equation for the four-potential

Substituting $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ into Maxwell's equations and using that partial derivatives commute,

$$\partial_\mu F^{\mu\nu} = \Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = -\mu_0 J^\nu,$$

that is,

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = -\mu_0 J^\nu.$$

Using the Lorenz gauge,

$$\partial_\mu A^\mu = 0,$$

the equations become

$$\boxed{\Box A^\nu = -\mu_0 J^\nu.}$$

3 Retarded Green function

The Green function satisfies

$$\Box G(x - x') = \delta^{(4)}(x - x').$$

The retarded solution is

$$G_R(x - x') = \frac{\Theta(x^0 - x'^0)\delta(x^0 - x'^0 - |\mathbf{x} - \mathbf{x}'|)}{4\pi|\mathbf{x} - \mathbf{x}'|}.$$

4 Current of a point charge

For a point charge,

$$J^\mu(x) = cq \int u^\mu(\tau)\delta^{(4)}(x - y(\tau))\,\mathrm{d}\tau,$$

where

- $y^\mu(\tau)$ is the worldline;
- $u^\mu = \mathrm{d}y^\mu/(c\,\mathrm{d}\tau)$ is the four-velocity, normalized as in the conventions.

5 Four-potential

Combining the Green function and the current,

$$A^\mu(x) = -\mu_0 \int G(x - z)J^\mu(z)\,\mathrm{d}^4 z,$$

Substituting the point-charge current and its own delta function collapses the z-integral onto the worldline, leaving a single integral over τ:

$$A^\mu(x) = \frac{\mu_0 q c^2}{4\pi} \int \frac{\Theta(x^0 - y^0(\tau))\delta(x^0 - y^0(\tau) - |\mathbf{x} - \mathbf{y}(\tau)|)}{|\mathbf{x} - \mathbf{y}(\tau)|} u^\mu(\tau)\,\mathrm{d}\tau.$$

Only values satisfying

$$x^0 - y^0(\tau) = |\mathbf{x} - \mathbf{y}(\tau)|$$

contribute to the integral.

6 Covariant form

Using

$$(x - y)^2 = -(x^0 - y^0)^2 + |\mathbf{x} - \mathbf{y}|^2,$$

and

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|},$$

one finds

$$\Theta(x^0 - y^0)\delta((x - y)^2) = \frac{\delta(x^0 - y^0 - |\mathbf{x} - \mathbf{y}|)}{2|\mathbf{x} - \mathbf{y}|}.$$

Therefore,

$$\boxed{A^\mu(x) = \frac{\mu_0 q c^2}{2\pi} \int \Theta(x^0 - y^0(\tau))\delta((x - y)^2)u^\mu(\tau)\,\mathrm{d}\tau.}$$

7 Retarded time

The retarded proper time is defined implicitly by

$$(x - y(\tau_r))^2 = 0,$$

with

$$x^0 > y^0(\tau_r).$$

Equivalently,

$$t - t_r = \frac{|\mathbf{x} - \mathbf{y}(t_r)|}{c}.$$

Write this as $f(t_r) = t - t_r - |\mathbf{x} - \mathbf{y}(t_r)|/c = 0$. Since $\mathrm{d}|\mathbf{x} - \mathbf{y}(t_r)|/\mathrm{d}t_r = -\hat{\mathbf{n}} \cdot \mathbf{v}$, with $\hat{\mathbf{n}}$ pointing from the charge to the field point, differentiating with respect to t_r gives

$$\dot{f} = \frac{\hat{\mathbf{n}} \cdot \mathbf{v}}{c} - 1 < 0,$$

which never vanishes, showing that the solution is unique.

Example (uniform velocity). For a charge moving with constant velocity $\mathbf{v}_0$, $\mathbf{y}(t_r) = \mathbf{y}_0 + \mathbf{v}_0 t_r$. Writing $\Delta\mathbf{x}_0 = \mathbf{x} - \mathbf{y}_0$, the defining condition $c(t - t_r) = |\mathbf{x} - \mathbf{y}(t_r)|$ squares to

$$c^2(t - t_r)^2 = |\Delta\mathbf{x}_0 - \mathbf{v}_0 t_r|^2 = |\Delta\mathbf{x}_0|^2 - 2\Delta\mathbf{x}_0 \cdot \mathbf{v}_0\, t_r + v_0^2 t_r^2.$$

Collecting powers of t_r and dividing by c^2 gives a quadratic equation,

$$\left(1 - \frac{v_0^2}{c^2}\right) t_r^2 - 2\left(t - \frac{\Delta\mathbf{x}_0 \cdot \mathbf{v}_0}{c^2}\right) t_r + \left(t^2 - \frac{|\Delta\mathbf{x}_0|^2}{c^2}\right) = 0,$$

with two roots — an advanced and a retarded one. Only the smaller root is causal ($t_r < t$), so

$$\boxed{t_r = \frac{\left(t - \frac{\Delta\mathbf{x}_0 \cdot \mathbf{v}_0}{c^2}\right) - \frac{1}{c}\sqrt{\left(ct - \frac{\Delta\mathbf{x}_0 \cdot \mathbf{v}_0}{c}\right)^2 - \left(1 - \frac{v_0^2}{c^2}\right)(c^2t^2 - |\Delta\mathbf{x}_0|^2)}}{1 - v_0^2/c^2}.}$$

8 Liénard–Wiechert potentials

It remains to evaluate the τ-integral in the boxed covariant potential using the same distributional identity as in the Covariant form section, now applied to $g(\tau) = (x - y(\tau))^2$, which has a single zero at $\tau = \tau_r$:

$$\delta((x - y(\tau))^2) = \frac{\delta(\tau - \tau_r)}{|g'(\tau_r)|}, \qquad g'(\tau) = -2c\, u(\tau) \cdot (x - y(\tau)),$$

where g' follows from $\mathrm{d}y^\mu/\mathrm{d}\tau = c\, u^\mu$. At $\tau = \tau_r$, with $\Delta x^\mu = x^\mu - y^\mu(\tau_r)$ having $\Delta x^0 = |\mathbf{x} - \mathbf{y}|$ and spatial part $|\mathbf{x} - \mathbf{y}|\hat{\mathbf{n}}$,

$$u \cdot \Delta x = -\gamma|\mathbf{x} - \mathbf{y}|\left(1 - \hat{\mathbf{n}} \cdot \mathbf{v}/c\right),$$

so that $|g'(\tau_r)| = 2c\gamma|\mathbf{x} - \mathbf{y}|(1 - \hat{\mathbf{n}} \cdot \mathbf{v}/c)$. The τ-integral then just picks out $u^\mu(\tau_r)/|g'(\tau_r)|$, and the factors of γ cancel against those hidden in $u^\mu = \gamma(1, \mathbf{v}/c)$, giving

$$\boxed{A^\mu = \frac{\mu_0 q c}{4\pi} \left.\frac{(1, \mathbf{v}/c)}{|\mathbf{x} - \mathbf{y}|\,(1 - \hat{\mathbf{n}} \cdot \mathbf{v}/c)}\right|_{t_r}.}$$

Hence

$$V = \frac{q}{4\pi\varepsilon_0} \left.\frac{1}{|\mathbf{x} - \mathbf{y}|(1 - \hat{\mathbf{n}} \cdot \mathbf{v}/c)}\right|_{t_r},$$

and

$$\mathbf{A} = \frac{\mu_0 q c}{4\pi} \left.\frac{\mathbf{v}}{|\mathbf{x} - \mathbf{y}|(1 - \hat{\mathbf{n}} \cdot \mathbf{v}/c)}\right|_{t_r}.$$

9 Derivatives of the potential

The field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ requires differentiating the covariant potential

$$A^\nu = \frac{\mu_0 q c^2}{2\pi} \int \Theta(x^0 - y^0(\tau))\delta((x - y(\tau))^2) u^\nu(\tau)\, \mathrm{d}\tau$$

with respect to x^μ. Only Θ and δ depend on x inside the integral (through $x - y(\tau)$), so

$$\partial_\mu A^\nu = \frac{\mu_0 q c^2}{2\pi} \int \Big[(\partial_\mu \Theta(x^0 - y^0))\delta((x-y)^2) + \Theta(x^0 - y^0)\, \partial_\mu[\delta((x-y)^2)]\Big] u^\nu \, \mathrm{d}\tau.$$

Discarding the self-field term. The first term has $\partial_\mu \Theta(x^0 - y^0) \propto \delta^0_\mu \, \delta(x^0 - y^0)$, which multiplies $\delta((x-y)^2)$. A point where both $x^0 = y^0$ and $(x-y)^2 = -(x^0 - y^0)^2 + |\mathbf{x} - \mathbf{y}|^2 = 0$ forces $\mathbf{x} = \mathbf{y}$ as well, i.e. $x = y$: this term is supported only at the charge's own location and gives an infinite self-field contribution there. It is dropped, keeping only the field at points away from the source:

$$\partial_\mu A^\nu = \frac{\mu_0 q c^2}{2\pi} \int \Theta(x^0 - y^0)\, \partial_\mu[\delta((x-y)^2)]\, u^\nu \, \mathrm{d}\tau.$$

Trading the x-derivative for a τ-derivative. By the chain rule,

$$\partial_\mu[\delta((x-y)^2)] = \delta'((x-y)^2)\, \partial_\mu (x-y)^2 = 2\Delta x_\mu \, \delta'((x-y)^2), \qquad \Delta x^\mu \equiv x^\mu - y^\mu(\tau).$$

Differentiating instead along the worldline, using $\mathrm{d}y^\mu/\mathrm{d}\tau = c\, u^\mu$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\delta((x-y)^2) = \delta'((x-y)^2)\, \frac{\mathrm{d}}{\mathrm{d}\tau}(x-y)^2 = \delta'((x-y)^2)\,(-2c\, u \cdot \Delta x),$$

so that

$$\delta'((x-y)^2) = -\frac{1}{2c\, u \cdot \Delta x} \frac{\mathrm{d}}{\mathrm{d}\tau}\delta((x-y)^2).$$

Substituting,

$$\partial_\mu A^\nu = -\frac{\mu_0 q c}{2\pi} \int \Theta(x^0 - y^0) \left[\frac{\mathrm{d}}{\mathrm{d}\tau}\delta((x-y)^2)\right] \frac{\Delta x_\mu \, u^\nu}{u \cdot \Delta x} \, \mathrm{d}\tau.$$

Integration by parts. Moving the τ-derivative off δ and onto the rest of the integrand costs a sign; the boundary term and the piece hitting Θ are, again, self-field contributions at $x = y$ and are dropped:

$$\partial_\mu A^\nu = \frac{\mu_0 q c}{2\pi} \int \Theta(x^0 - y^0)\, \delta((x-y)^2)\, \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\Delta x_\mu \, u^\nu}{u \cdot \Delta x}\right) \mathrm{d}\tau.$$

The τ-integral is now done exactly as in the Liénard–Wiechert derivation, using $\delta((x-y)^2) = \delta(\tau - \tau_r)/(2c|u \cdot \Delta x|)$:

$$\partial_\mu A^\nu = \frac{\mu_0 q}{4\pi} \frac{1}{|u \cdot \Delta x|} \, \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\Delta x_\mu \, u^\nu}{u \cdot \Delta x}\right)\Bigg|_{\tau = \tau_r}.$$

Differentiating term by term. Writing the four-acceleration as $a^\mu = \mathrm{d}u^\mu/\mathrm{d}\tau$, the product rule gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\Delta x_\mu \, u^\nu}{u \cdot \Delta x}\right) = \frac{(\dot{\Delta x_\mu})\, u^\nu + \Delta x_\mu \, a^\nu}{u \cdot \Delta x} - \frac{\Delta x_\mu \, u^\nu}{(u \cdot \Delta x)^2} \frac{\mathrm{d}}{\mathrm{d}\tau}(u \cdot \Delta x).$$

Since x is fixed, $(\dot{\Delta x_\mu}) = -\mathrm{d}y_\mu/\mathrm{d}\tau = -c\, u_\mu$, and

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(u \cdot \Delta x) = a \cdot \Delta x + u \cdot (\dot{\Delta x}) = a \cdot \Delta x - c\, u \cdot u = a \cdot \Delta x + c,$$

using $u \cdot u = -1$. Substituting both,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\Delta x_\mu \, u^\nu}{u \cdot \Delta x}\right) = \frac{-c\, u_\mu u^\nu + \Delta x_\mu \, a^\nu}{u \cdot \Delta x} - \frac{\Delta x_\mu \, u^\nu}{(u \cdot \Delta x)^2}(\Delta x \cdot a + c),$$

so that

$$\partial_\mu A^\nu = \frac{\mu_0 q}{4\pi} \frac{1}{(u \cdot \Delta x)^2} \left[-c\, u_\mu u^\nu + \Delta x_\mu \, a^\nu - \frac{\Delta x_\mu \, u^\nu}{u \cdot \Delta x}(\Delta x \cdot a + c)\right]_{\tau = \tau_r}.$$

The term $-c\,u_\mu u^\nu$ is symmetric under $\mu \leftrightarrow \nu$ (raising the index on μ), so it cancels in the antisymmetric combination $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and can be dropped when only the field, not the potential's derivative itself, is wanted:

$$\boxed{\partial_\mu A_\nu = \frac{\mu_0 q}{4\pi}\frac{1}{(u\cdot\Delta x)^2}\left[\Delta x_\mu\, a_\nu - \frac{\Delta x_\mu\, u_\nu}{u\cdot\Delta x}(\Delta x\cdot a + c)\right]_{\tau=\tau_r} + \text{ (symmetric in } \mu\nu).}$$

Near field versus radiation field. Both $u\cdot\Delta x$ and Δx_μ scale with the distance $R = |\mathbf{x}-\mathbf{y}|$ to the source: $u\cdot\Delta x \sim R$ (Liénard–Wiechert section) and $\Delta x_\mu \sim R$, while u^ν and a^ν, evaluated at the fixed retarded time τ_r, do not depend on R. In the boxed expression, the term $\Delta x_\mu a_\nu/(u\cdot\Delta x)^2$ scales as $R/R^2 = 1/R$: it carries the four-acceleration explicitly, and is the **radiation field**. In the second term, the factor $\Delta x\cdot a$ inside the parenthesis itself grows as R, so it contributes another $R/(u\cdot\Delta x)^3 \sim 1/R$ piece to the radiation field, while the leftover c — the only acceleration-independent piece left in the whole expression — contributes $\Delta x_\mu u_\nu c/(u\cdot\Delta x)^3 \sim 1/R^2$. This is exactly the statement in the original notes: **terms carrying the acceleration fall off as** $1/R$; whatever survives without it falls off as $1/R^2$ and is the near (velocity) field.

10 The radiation field

The boxed formula above is for $\partial_\mu A_\nu$, not yet the field. Since the symmetric piece cancels in the antisymmetric combination, the full field tensor is obtained simply by antisymmetrizing the whole bracket (dropping nothing this time — this is exact, not just the radiative part):

$$F_{\mu\nu} = \frac{\mu_0 q}{4\pi(u\cdot\Delta x)^2}\left[\Delta x_\mu a_\nu - \Delta x_\nu a_\mu - \frac{\Delta x\cdot a + c}{u\cdot\Delta x}(\Delta x_\mu u_\nu - \Delta x_\nu u_\mu)\right]_{\tau=\tau_r}.$$

Splitting off the radiative ($1/R$) part identified above — the terms carrying a^μ — from the near-field ($1/R^2$) part left over,

$$F_{\mu\nu} = \underbrace{\frac{\mu_0 q}{4\pi(u\cdot\Delta x)^2}\left[\Delta x_\mu a_\nu - \Delta x_\nu a_\mu - \frac{\Delta x\cdot a}{u\cdot\Delta x}(\Delta x_\mu u_\nu - \Delta x_\nu u_\mu)\right]}_{F^{\rm rad}_{\mu\nu}} - \underbrace{\frac{\mu_0 q c}{4\pi(u\cdot\Delta x)^3}(\Delta x_\mu u_\nu - \Delta x_\nu u_\mu)}_{F^{\rm near}_{\mu\nu}}.$$

Why the radiation field is transverse. Contract $F^{\rm rad}_{\mu\nu}$ with Δx^ν. Every term picks up either $\Delta x\cdot\Delta x$ or a combination that cancels exactly:

$$F^{\rm rad}_{\mu\nu}\Delta x^\nu = \frac{\mu_0 q}{4\pi(u\cdot\Delta x)^2}\Big[\Delta x_\mu(a\cdot\Delta x) - a_\mu(\Delta x\cdot\Delta x) - \frac{\Delta x\cdot a}{u\cdot\Delta x}(\Delta x_\mu(u\cdot\Delta x) - u_\mu(\Delta x\cdot\Delta x))\Big].$$

Since $\Delta x = x - y(\tau_r)$ is null, $\Delta x\cdot\Delta x = 0$ (the defining condition of τ_r), so this collapses to

$$F^{\rm rad}_{\mu\nu}\Delta x^\nu = \frac{\mu_0 q}{4\pi(u\cdot\Delta x)^2}\Big[\Delta x_\mu(a\cdot\Delta x) - \Delta x_\mu(a\cdot\Delta x)\Big] = 0.$$

The same is not true of $F^{\rm near}_{\mu\nu}$ (there is nothing to cancel the leftover $u_\mu(\Delta x\cdot\Delta x)$ term against): only the radiation field is null in this sense, not the full field.

In 3-vector form. With $\Delta x^0 = R \equiv |\mathbf{x}-\mathbf{y}|$, $\Delta x^i = R\hat{n}^i$, and $F^{0i} = E^i/c$, $F^{ij} = \epsilon^{ijk}B_k$, the statement $F^{\rm rad}_{\mu\nu}\Delta x^\nu = 0$ reads, for $\mu = 0$,

$$F^{\rm rad}_{0i}\Delta x^i = 0 \implies \hat{\mathbf{n}}\cdot\mathbf{E}_{\rm rad} = 0,$$

and for $\mu = j$, using $F_{j0} = E_j/c$ and $F_{ji} = \epsilon_{jik}B_k$,

$$F^{\rm rad}_{j0}\Delta x^0 + F^{\rm rad}_{ji}\Delta x^i = 0 \implies \frac{E_j}{c} = -\epsilon_{jik}\hat{n}^i B_k \implies \boxed{\mathbf{B}_{\rm rad} = \frac{1}{c}\,\hat{\mathbf{n}}\times\mathbf{E}_{\rm rad}, \qquad \hat{\mathbf{n}}\cdot\mathbf{E}_{\rm rad} = 0.}$$

This is exact, for a source moving with any velocity: the radiation field looks locally like a plane wave propagating along $\hat{\mathbf{n}}$, which is why $|\mathbf{E}_{\text{rad}}| = c|\mathbf{B}_{\text{rad}}|$ is used below. The explicit radiation field itself follows from $E_i^{\text{rad}} = -cF_{0i}^{\text{rad}}$,

$$\mathbf{E}_{\text{rad}} \cdot \hat{\mathbf{e}}_i = -\frac{\mu_0 q c}{4\pi(u \cdot \Delta x)^2} \left[\Delta x_0 a_i - \Delta x_i a_0 - \frac{\Delta x \cdot a}{u \cdot \Delta x} (\Delta x_0 u_i - \Delta x_i u_0) \right]_{\tau=\tau_r},$$

valid for any velocity of the source. Specializing to a source instantaneously at rest, $u^\mu = (1, \mathbf{0})$, reduces this to the non-relativistic dipole field used in Thomson Scattering.

11 Radiation zone

Far from the source only the radiation field survives, with $B = E/c$ as just shown. Substituting this into the energy density and Poynting vector from Energy of the Electromagnetic Field,

$$u = \frac{1}{2\mu_0} \left(\frac{E^2}{c^2} + B^2 \right), \qquad \mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B},$$

gives

$$\boxed{u = \varepsilon_0 E^2, \qquad \mathbf{S} = \frac{E^2}{\mu_0 c} \hat{\mathbf{n}} = cu\,\hat{\mathbf{n}}.}$$

Energy flows radially outward at the speed of light, as expected for radiation. (Here u is the field energy density, unrelated to the four-velocity u^μ of the previous sections; both are standard uses of the same letter and are always distinguished by the presence of the index.)

12 Covariant radiation tensor

Write $F_{\mu\nu}^{\text{rad}} = C(\Delta x_\mu b_\nu - \Delta x_\nu b_\mu)$, with $C \equiv \mu_0 q/[4\pi(u \cdot \Delta x)^2]$ and $b_\nu \equiv a_\nu - \lambda u_\nu$, $\lambda \equiv (\Delta x \cdot a)/(u \cdot \Delta x)$, matching the bracket in the formula above term by term. Two facts make what follows easy:

$$\Delta x \cdot b = \Delta x \cdot a - \lambda(\Delta x \cdot u) = \Delta x \cdot a - \Delta x \cdot a = 0, \qquad \Delta x \cdot \Delta x = 0.$$

The first is just λ's definition; the second is the null condition used above. Together they make the field itself null,

$$F_{\text{rad}}^{\alpha\beta} F^{\text{rad}}_{\alpha\beta} = 2C^2 \left[(\Delta x \cdot \Delta x)(b \cdot b) - (\Delta x \cdot b)^2 \right] = 0,$$

consistent with $|\mathbf{E}_{\text{rad}}| = c|\mathbf{B}_{\text{rad}}|$ found above (a plane wave has $F^{\alpha\beta} F_{\alpha\beta} = 2(B^2 - E^2/c^2) = 0$ exactly when $E = cB$). So the second term of $T^{\mu\nu} = \frac{1}{\mu_0}(F^{\mu\alpha} F^\nu{}_\alpha - \frac{1}{4}\eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta})$ drops out for the radiation piece, leaving

$$\mu_0 T^{\mu\nu}_{\text{rad}} = F^{\mu\alpha}_{\text{rad}} F^\nu{}_{\alpha,\text{rad}} = C^2 (\Delta x^\mu b^\alpha - \Delta x^\alpha b^\mu)(\Delta x^\nu b_\alpha - \Delta x_\alpha b^\nu).$$

Expanding the product, every term with a lone $\Delta x \cdot b$ or $\Delta x \cdot \Delta x$ vanishes, leaving only

$$\mu_0 T^{\mu\nu}_{\text{rad}} = C^2 (b \cdot b)\, \Delta x^\mu \Delta x^\nu.$$

Finally, expand $b \cdot b = (a - \lambda u) \cdot (a - \lambda u) = a \cdot a - 2\lambda(a \cdot u) + \lambda^2 (u \cdot u)$. The middle term drops since $a \cdot u = 0$, and $u \cdot u = -1$ turns the last term negative, so

$$b \cdot b = a \cdot a - \lambda^2 = \frac{(a \cdot a)(\Delta x \cdot u)^2 - (\Delta x \cdot a)^2}{(\Delta x \cdot u)^2}.$$

Substituting $C^2 = \mu_0^2 q^2 / [16\pi^2 (u \cdot \Delta x)^4]$,

$$\boxed{T^{\mu\nu}_{\text{rad}} = \frac{\mu_0 q^2}{16\pi^2 (\Delta x \cdot u)^6} \left[(a \cdot a)(\Delta x \cdot u)^2 - (\Delta x \cdot a)^2 \right] \Delta x^\mu \Delta x^\nu.}$$

Important observations:

- proportional to $\Delta x^\mu \Delta x^\nu$;
- null, since $\Delta x^2 = 0$ on the light cone;
- directed along outgoing light rays;
- survives at arbitrarily large distances, since the $1/|\Delta \mathbf{x}|^2$ falloff exactly matches the near-field terms falling off faster.

This is the field responsible for energy loss by the source.

13 Relativistic Larmor formula

"Power radiated" is ambiguous for an accelerated source unless a frame is specified: a general observer sees the flux Doppler-shifted by the retarded-time relation between source and observation time. The unambiguous, covariant quantity is the energy radiated **per unit proper time of the source**, computed as the flux of $T^{\mu\nu}_{\text{rad}}$ through a sphere surrounding the charge *in its own instantaneous rest frame*, where this subtlety is absent because the source is momentarily not moving. Since this power, so defined, and $a^\mu a_\mu$ are both Lorentz scalars, a relation between them established in one frame (the simplest one, the rest frame) holds in every frame.

Step 1: compute the flux in the rest frame. There, $u^\mu = (1, \mathbf{0})$, $\Delta x \cdot u = -R$, and $u_\mu a^\mu = 0$ forces $a^\mu = (0, \dot{\mathbf{v}}/c)$ for the ordinary acceleration $\dot{\mathbf{v}} = \mathrm{d}\mathbf{v}/\mathrm{d}t$, so

$$a \cdot a = \frac{\dot{v}^2}{c^2}, \qquad \Delta x \cdot a = R\,\frac{\hat{\mathbf{n}} \cdot \dot{\mathbf{v}}}{c},$$

exactly as used for the Thomson-scattering radiation field. Substituting into the boxed $T^{\mu\nu}_{\text{rad}}$ above, with χ the angle between $\hat{\mathbf{n}}$ and $\dot{\mathbf{v}}$ (kept distinct from the Heaviside Θ used throughout this page),

$$(a \cdot a)(\Delta x \cdot u)^2 - (\Delta x \cdot a)^2 = \frac{R^2}{c^2}\left[\dot{v}^2 - (\hat{\mathbf{n}} \cdot \dot{\mathbf{v}})^2\right] = \frac{R^2 \dot{v}^2}{c^2} \sin^2 \chi,$$

so that, using $T^{0i} = S^i/c$ and $(\Delta x \cdot u)^6 = R^6$,

$$\mathbf{S} \cdot \hat{\mathbf{n}} = cT^{0i}\hat{n}^i = c \cdot \frac{\mu_0 q^2}{16\pi^2 R^6} \cdot \frac{R^2 \dot{v}^2 \sin^2 \chi}{c^2} \cdot R^2 = \frac{\mu_0 q^2 \dot{v}^2}{16\pi^2 c R^2} \sin^2 \chi.$$

Step 2: integrate over the sphere. The power radiated per solid angle is

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = R^2\, \mathbf{S} \cdot \hat{\mathbf{n}} = \frac{\mu_0 q^2 \dot{v}^2}{16\pi^2 c} \sin^2 \chi.$$

Using $\int \sin^2 \chi \, \mathrm{d}\Omega = 2\pi \int_0^\pi \sin^3 \chi \, \mathrm{d}\chi = 2\pi \cdot \frac{4}{3} = \frac{8\pi}{3}$,

$$P = \int \frac{\mathrm{d}P}{\mathrm{d}\Omega}\, \mathrm{d}\Omega = \frac{\mu_0 q^2 \dot{v}^2}{16\pi^2 c} \cdot \frac{8\pi}{3} = \frac{\mu_0 q^2 \dot{v}^2}{6\pi c},$$

the familiar non-relativistic Larmor formula.

Step 3: covariantize. In this same rest frame, $a^\mu a_\mu = \dot{v}^2/c^2$, so the result of Step 2 is exactly

$$P = \frac{\mu_0 q^2 c}{6\pi}\, a^\mu a_\mu.$$

Both sides of this equation are Lorentz scalars (P by the proper-time construction above, $a^\mu a_\mu$ trivially), and they agree in one frame; therefore they agree in every frame. The relation is thus valid for a source moving with arbitrary velocity, not just at the instant it is at rest:

$$\boxed{P = \frac{\mu_0 q^2 c}{6\pi} a^\mu a_\mu.}$$

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