

# Maxwell Equations in Covariant Form

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## Scope

These notes build on the calculus tools developed previously and show how Maxwell equations naturally unify into a single geometric statement on spacetime.

The key steps are:

- recognizing that  $\mathbf{E}$  and  $\mathbf{B}$  are observer-dependent projections,
- combining them into the Faraday tensor  $F_{\mu\nu}$ ,
- writing homogeneous and non-homogeneous Maxwell equations covariantly,
- understanding the geometric role of differential forms in this framework.

## From 3D Vector Calculus to Covariant Electromagnetism

### Motivation

Classical electromagnetism is usually formulated with a preferred time slicing, where fields are split into spatial vectors  $\mathbf{E}$  and  $\mathbf{B}$ . This formulation is not covariant: under Lorentz transformations, time mixes with space, and electric and magnetic fields mix with each other.

This suggests that  $\mathbf{E}$  and  $\mathbf{B}$  are not fundamental objects. Instead, they should arise as components of a single spacetime object.

Our goal is to rewrite Maxwell equations in a form that:

- does not depend on a preferred frame,
- is manifestly Lorentz covariant,
- has a clear geometric meaning.

### Homogeneous vs Non-Homogeneous Equations

Maxwell equations naturally split into two groups:

**Non-homogeneous** (contain sources):

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \quad \nabla \times \mathbf{B} - \frac{1}{c^2} \partial_t \mathbf{E} = \mu_0 \mathbf{J},$$

where  $\rho$  is the charge density,  $\mathbf{J}$  is the current density, and  $\varepsilon_0, \mu_0$  are the vacuum permittivity and permeability. Finally,  $c = 1/\sqrt{\varepsilon_0 \mu_0}$  is the speed of light.

**Homogeneous** (purely geometric):

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0.$$

The homogeneous equations do not depend on sources. This suggests they encode a purely geometric structure.

## Hidden Metric in Vector Calculus

Operators like gradient, divergence, and curl implicitly use the metric. For instance, the gradient is not just  $\partial_i$ , but effectively:

$$(\nabla f)^i = g^{ij} \partial_j f.$$

This becomes important when changing coordinates or working covariantly. The form language makes these dependencies explicit.

## From Vectors to Forms

To make the geometry explicit, we reinterpret fields as differential forms:

- Electric field  $\mathbf{E}$  becomes a 1-form:

$$E = E_i dx^i,$$

- Magnetic field  $\mathbf{B}$  becomes a 2-form:

$$B = \frac{1}{2} B_{ij} dx^i \wedge dx^j.$$

The magnetic field is naturally a 2-form because it is associated with flux through surfaces, which are 2-dimensional.

## Unification into a Single Spacetime Equation

### The Key Observation

The homogeneous equations:

- Faraday law:  $\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0$ ,
- No magnetic monopoles:  $\nabla \cdot \mathbf{B} = 0$ ,

are actually different components of a single spacetime equation.

### Spacetime Coordinates

Introduce spacetime coordinates with signature  $(-, +, +, +)$ :

$$x^0 = ct, \quad x^i \quad (i = 1, 2, 3) \text{ spatial.}$$

### The Electromagnetic Field Tensor

Define the **Faraday tensor**  $F_{\mu\nu}$  as a 2-form on spacetime with components:

$$F_{0i} = E_i, \quad F_{ij} = B_{ij},$$

and antisymmetry  $F_{\mu\nu} = -F_{\nu\mu}$ .

In matrix form (with  $c = 1$ ):

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}.$$

### The Homogeneous Maxwell Equation

The homogeneous Maxwell equations become:

$$dF = 0.$$

This is a single equation on spacetime, encoding all four components:

- the 3 spatial components of Faraday's law,
- the 1 component of  $\nabla \cdot \mathbf{B} = 0$ .

## Verification in Components

The equation  $dF = 0$  expands to:

$$\partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} = 0.$$

For  $(\alpha, \mu, \nu) = (0, i, j)$ :

$$\partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} = 0,$$

which gives:

$$\partial_t B_{ij} - \partial_i E_j + \partial_j E_i = 0,$$

exactly Faraday's law.

For  $(\alpha, \mu, \nu) = (i, j, k)$  (cyclic):

$$\partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij} = 0,$$

which is equivalent to  $\nabla \cdot \mathbf{B} = 0$ .

## Geometric Interpretation

### Form Degree and Dimension

- $F$  is a 2-form on spacetime (6 independent components),
- $dF$  is a 3-form on spacetime (4 independent components in 4D).

In 4 dimensions, a 3-form has exactly

$$\binom{4}{3} = 4$$

independent components. These correspond to:

- the 3 components of Faraday's law,
- the 1 component of  $\nabla \cdot \mathbf{B} = 0$ .

Thus, covariance naturally combines them into a single geometric statement.

### Closure Condition

The identity  $d^2 = 0$  guarantees consistency. If  $F$  is itself exact,  $F = dA$ , then:

$$dF = d^2A = 0$$

automatically. This motivates the introduction of the electromagnetic potential  $A_\mu$ .

## The Non-Homogeneous Equations

### Source Term

The non-homogeneous equations involve the current 4-vector:

$$J^\mu = (\rho, J^x, J^y, J^z).$$

They are written covariantly as:

$$\partial_\mu F^{\mu\nu} = J^\nu,$$

or equivalently, using the Hodge dual:

$$d \star F = \star J.$$

## Component Check

For  $\nu = 0$ :

$$\partial_i F^{i0} = J^0 = \rho,$$

which gives Gauss's law  $\nabla \cdot \mathbf{E} = \rho$ .

For  $\nu = k$  (spatial):

$$\partial_0 F^{0k} + \partial_i F^{ik} = J^k,$$

which expands to:

$$\partial_t E^k + (\nabla \times \mathbf{B})^k = J^k,$$

the Ampère-Maxwell law.

## Final Structure

Electromagnetism in covariant form:

### ! Maxwell Equations

**Field strength:**

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

**Homogeneous equations:**

$$dF = 0.$$

**Non-homogeneous equations:**

$$\partial_\mu F^{\mu\nu} = J^\nu.$$

## Conceptual Summary

- The split into  $\mathbf{E}$  and  $\mathbf{B}$  is frame-dependent,
- Vector calculus hides the underlying spacetime geometry,
- Differential forms make the covariant structure explicit,
- Maxwell equations reduce to simple geometric statements on spacetime.

This formulation prepares the ground for:

- understanding gauge freedom ( $A_\mu \rightarrow A_\mu + \partial_\mu \chi$ ),
- generalizing to curved spacetime (general relativity),
- interpreting electromagnetism as a  $U(1)$  gauge theory.

## Exercises

1. Verify that the matrix representation of  $F_{\mu\nu}$  given above is antisymmetric and correctly encodes  $\mathbf{E}$  and  $\mathbf{B}$ .
2. Show explicitly that  $dF = 0$  gives the component  $\partial_t B_{12} - \partial_1 E_2 + \partial_2 E_1 = 0$ , and identify this with one component of Faraday's law.
3. Prove that if  $F = dA$ , then  $dF = 0$  automatically.
4. Verify that  $\partial_\mu F^{\mu 0} = \rho$  reproduces Gauss's law when  $F^{i0} = -E^i$ .
5. Show that the gauge transformation  $A_\mu \rightarrow A_\mu + \partial_\mu \chi$  leaves  $F_{\mu\nu}$  unchanged.
6. Write the action for free electromagnetism:

$$S = -\frac{1}{4} \int F_{\mu\nu} F^{\mu\nu} d^4x,$$

and derive the vacuum Maxwell equations by varying  $A_\mu$ .

7. In 2+1 dimensions, how many independent components does  $F_{\mu\nu}$  have? How many for  $dF$ ?
8. Discuss why the Hodge dual maps a 2-form to another 2-form in 4D, and explain its role in the source equations.