Geometry Review for Electrodynamics

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Conventions and Notation

- Signature: $(-,+,+,+)$ (convention)
- Greek indices: $\mu, \nu = 0, 1, 2, 3$
- Einstein summation implied
- Time coordinate: $x^0 = ct$

Metric:

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$$

The choice of signature is a convention: one may equivalently use $(+,-,-,-)$ provided it is applied consistently. Physical predictions are independent of this choice.

Instead of setting $c = 1$, we define the time coordinate as $x^0 = ct$, so that all components of x^μ have dimensions of length. In this formulation,

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

which is algebraically equivalent to working in units with $c = 1$. The two approaches differ only by a rescaling of the time coordinate.

Relativistic Framework

Motivation and Space-Time Structure

- Postulates of special relativity
- Lorentz invariance
- Space-time as a unified geometric object
- Worldlines

The postulates (constancy of c and equivalence of inertial frames) require a unified treatment of space and time. Particle histories are described by worldlines, making causal relations explicit.

Lorentz transformations:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

Minkowski Space

- Vector space $\mathbb{R}^4$ with indefinite metric

Interval:

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$$

Invariance:

$$\mathrm{d}s'^2 = \mathrm{d}s^2$$

The invariant interval replaces separate notions of spatial distance and time separation.

Euclidean Comparison

- Positive-definite inner product

$$\|\mathbf{a}\|^2 = \mathbf{a} \cdot \mathbf{a}$$

Inequalities:

$$|\mathbf{a} \cdot \mathbf{b}| \le \|\mathbf{a}\|\|\mathbf{b}\|, \quad \|\mathbf{a} + \mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\|$$

These properties rely on positive definiteness and do not directly extend to Minkowski space.

Minkowski Geometry and Causality

Norm:

$$A^\mu A_\mu = \eta_{\mu\nu} A^\mu A^\nu$$

Classification:

- $A^\mu A_\mu < 0$ — timelike
- $A^\mu A_\mu > 0$ — spacelike
- $A^\mu A_\mu = 0$ — null

Light cone:

$$ds^2 = 0$$

Regions:

- future: $x^0 > 0$, $ds^2 < 0$
- past: $x^0 < 0$, $ds^2 < 0$
- spacelike: $ds^2 > 0$

Timelike separation allows causal influence; spacelike does not.

Proper Time

Definition:

$$d\tau^2 = -ds^2$$

Worldline:

$$au = \int \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu}$$

Proper time is invariant and depends on the trajectory.

Lorentz Transformations

Definition:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

Constraint:

$$\eta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu = \eta_{\mu\nu}$$

Boost (along x):

$$t' = \gamma(t - vx)$$
$$x' = \gamma(x - vt)$$

Four-Vectors

Position:

$$x^\mu = (t, x, y, z)$$

Four-velocity:

$$u^\mu = \frac{dx^\mu}{d\tau}, \quad u^\mu u_\mu = -1$$

Momentum:

$$p^\mu = m u^\mu$$

Minkowski Inner Product and Inequalities

Definition:

$$A^\mu B_\mu = \eta_{\mu\nu} A^\mu B^\nu$$

Lorentz invariant.

For timelike, same time orientation:

$$(A^\mu B_\mu)^2 \geq (A^\mu A_\mu)(B^\nu B_\nu)$$

Inequalities require causal restrictions.

Tensor and Differential Structure

Index Notation

- Contravariant: A^μ
- Covariant: A_μ

Lowering:

$$A_\mu = \eta_{\mu\nu} A^\nu$$

Raising:

$$A^\mu = \eta^{\mu\nu} A_\nu$$

Vectors and Covectors

- Tangent space: V
- Dual space: V^*

Pairing:

$$\omega(v)$$

Gradient:

$$\partial_\mu$$

Covectors act linearly on vectors; ∂_μ is naturally covariant.

Metric as Isomorphism

Mapping:

$$v^\mu \leftrightarrow v_\mu$$

The metric identifies V and V^* and defines contractions.

Tensors

Transformation:

$$T'^{\mu}{}_{\nu} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\beta}{}_{\nu} T^{\alpha}{}_{\beta}$$

Tensor equations with fully contracted indices are invariant.

Differential Operators

Four-gradient:

$$\partial_\mu$$

d'Alembertian:

$$\Box = \partial_\mu \partial^\mu$$

Lorentz-invariant wave operator.

Differential Forms and Electrodynamics

Differential Forms

Basis:

$$\mathrm{d}x^{\mu}$$

1-form:

$$\omega = \omega_{\mu} \mathrm{d}x^{\mu}$$

Exterior product:

$$\mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} = -\mathrm{d}x^{\nu} \wedge \mathrm{d}x^{\mu}$$

Electromagnetic 2-Form

$$F = \frac{1}{2} F_{\mu\nu} \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu}$$

Encodes electric and magnetic fields covariantly.

Hodge Dual

Mapping:

$$\star : \Omega^{p} \to \Omega^{4-p}$$

Property:

$$\star \star \omega = (-1)^{p(4-p)+1} \omega$$

Components:

$$(\star F)_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$$

Integration and Orientation

- Integration of p-forms
- Orientation

Volume element:

$$\mathrm{d}^4 x$$

Defines invariant integration measures.

Stokes' Theorem

$$\int_{\partial M} \omega = \int_{M} \mathrm{d}\omega$$

Unifies classical integral theorems.

Covariant Electrodynamics

Potential:

$$A = A_\mu \mathrm{d}x^\mu$$

Field:

$$F = \mathrm{d}A$$

Components:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Gauge symmetry:

$$A \mapsto A + \mathrm{d}\lambda$$

Maxwell Equations

Homogeneous:

$$\mathrm{d}F = 0$$

With sources:

$$\mathrm{d} \star F = J$$

Integral Laws

Gauss:

$$\int_{\partial V} \star F = \int_V J$$

Faraday:

$$\int_{\partial S} F = 0$$

Ampère–Maxwell:

$$\int_{\partial S} \star F = \int_S J$$

Charge Conservation

$$\partial_\mu J^\mu = 0$$

Follows from:

$$\mathrm{d}(\mathrm{d} \star F) = 0 \Rightarrow \mathrm{d}J = 0$$

Structural Summary

Geometric Chain

- Metric $\to$ inner product
- Inner product $\to$ duality
- Duality $\to$ Hodge star
- d and $\star$ $\to$ Maxwell equations
- Stokes $\to$ integral laws

Exercises

Conceptual and Computational

1. Classify intervals (timelike, spacelike, null) with explicit examples.
2. Verify invariance of ds^2 under a boost.
3. Compute $F_{\mu\nu}$ from a given A_μ and identify $\mathbf{E}$ and $\mathbf{B}$.
4. Show $dF = 0$ in components and relate to Faraday's law.
5. Derive Gauss' law from $d \star F = J$.
6. Compute $\star F$ for a plane wave and interpret components.