Energy of the Electromagnetic Field

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These notes follow the conventions fixed earlier in the course: SI units, signature $(-,+,+,+)$, and $x^\mu = (ct, \mathbf{x})$.

1 Energy stored in the electrostatic field

Assembling a charge distribution means repeatedly bringing in a small charge $\mathrm{d}q$ from infinity against the potential V already set up by the charge in place, doing work

$$\mathrm{d}W = V\,\mathrm{d}q,$$

and choosing the reference such that the potential vanishes at infinity,

$$V(\infty) = 0.$$

Summing this over the whole distribution ρ, with the factor $\frac{1}{2}$ compensating for counting each pair of charges twice, the total electrostatic energy is

$$W = \frac{1}{2}\int_V V\rho\,\mathrm{d}^3x.$$

Using Poisson's equation,

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0},$$

we obtain

$$W = -\frac{\varepsilon_0}{2}\int_V V\nabla^2 V\,\mathrm{d}^3x.$$

Trading ρ for a total-derivative term plus $|\nabla V|^2$ with the product rule,

$$V\nabla^2 V = \nabla\cdot(V\nabla V) - |\nabla V|^2,$$

and applying the divergence theorem to the first piece gives

$$W = -\frac{\varepsilon_0}{2}\oint_{\partial V} V\nabla V\cdot\mathrm{d}\mathbf{a} + \frac{\varepsilon_0}{2}\int_V |\mathbf{E}|^2\,\mathrm{d}^3x.$$

Assuming the surface term vanishes,

$$\boxed{W = \frac{\varepsilon_0}{2}\int_V |\mathbf{E}|^2\,\mathrm{d}^3x.}$$

2 Building a stationary current

After assembling the electrostatic configuration we slowly establish a stationary current. During this process

- the charge distribution is stationary,
- $\partial\mathbf{E}/\partial t = 0$ for the electrostatic field,
- the current varies with time,
- therefore the magnetic field also varies.

The induced electric field satisfies Faraday's law,

$$\nabla \times \mathbf{E}_c = -\frac{\partial \mathbf{B}}{\partial t}.$$

Here $\mathbf{E}_c$ denotes the electric field induced by the changing magnetic field.

3 Quasi-static approximation

Ampère–Maxwell's equation reads

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}_c}{\partial t} = \mu_0 \mathbf{J}.$$

Estimating $\nabla \sim 1/L$ for a source of size L, Ampère's law with the displacement current dropped gives $B \sim \mu_0 J L$,

$$\nabla \sim \frac{1}{L}, \qquad B \sim \mu_0 J L,$$

and Faraday's law, with T the timescale over which the current changes, gives in the same way $E_c \sim LB/T$,

$$E_c \sim \frac{LB}{T},$$

and therefore

$$\frac{1}{c^2}\frac{\partial E_c}{\partial t} \sim \mu_0 J \left(\frac{L}{cT}\right)^2.$$

Hence, if

$$\boxed{\frac{L}{T} \ll c,}$$

the displacement current can be neglected.

4 Magnetic energy

For a single charge, the Lorentz force gives the power delivered by the fields as $\mathrm{d}W/\mathrm{d}t = \mathbf{v}\cdot\mathbf{F} = \mathbf{v}\cdot(q\mathbf{E}_c + q\mathbf{v}\times\mathbf{B}) = q\mathbf{v}\cdot\mathbf{E}_c$, since the magnetic force is always perpendicular to $\mathbf{v}$ and does no work. For a continuous distribution, $q\mathbf{v} \to \mathbf{J}\,\mathrm{d}^3x$, so the work done on the charges is

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \int \mathbf{J}\cdot\mathbf{E}_c\,\mathrm{d}^3x.$$

Using the quasi-static approximation,

$$\mathbf{J} = \frac{1}{\mu_0}\nabla\times\mathbf{B},$$

and the identity

$$\nabla\cdot(\mathbf{E}_c\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{E}_c) - \mathbf{E}_c\cdot(\nabla\times\mathbf{B}),$$

together with Faraday's law,

$$\nabla \times \mathbf{E}_c = -\frac{\partial \mathbf{B}}{\partial t},$$

the integrand becomes a divergence plus $-\mathbf{B}\cdot\partial_t\mathbf{B}/\mu_0 = -\frac{1}{2\mu_0}\partial_t|\mathbf{B}|^2$, so that, assuming the surface term vanishes,

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\frac{1}{2\mu_0}\frac{\partial}{\partial t}\int |\mathbf{B}|^2\,\mathrm{d}^3x.$$

This is negative: the induced field $\mathbf{E}_c$ opposes the growing current (Lenz's law), so it does negative work on the charges, and whatever drives the current against $\mathbf{E}_c$ must supply the balance. That energy, $W_B \equiv -W$, ends up stored in the magnetic field; if the field is initially zero,

$$\boxed{W_B = \frac{1}{2\mu_0}\int |\mathbf{B}|^2\,\mathrm{d}^3x.}$$

5 Total electromagnetic energy

Once the current is steady, $\mathbf{E}_c$ has done its job and vanishes; what is left is the electrostatic field of the assembled charges plus the magnetic field of the assembled current, and the configuration “remembers” the energy spent building it: $U \equiv W + W_B$, the total field energy, is

$$\boxed{U = \frac{1}{2\mu_0} \int \left(\frac{|\mathbf{E}|^2}{c^2} + |\mathbf{B}|^2 \right) \mathrm{d}^3x.}$$

Defining the energy density,

$$u = \frac{1}{2\mu_0} \left(\frac{|\mathbf{E}|^2}{c^2} + |\mathbf{B}|^2 \right),$$

its time derivative is

$$\frac{\partial u}{\partial t} = \frac{1}{\mu_0} \left(\frac{\mathbf{E} \cdot \partial_t \mathbf{E}}{c^2} + \mathbf{B} \cdot \partial_t \mathbf{B} \right).$$

Now, unlike in the magnetic-energy calculation, the source $\mathbf{J}$ is kept: solving Ampère–Maxwell’s law for the displacement current, $\partial_t \mathbf{E}/c^2 = \nabla \times \mathbf{B} - \mu_0 \mathbf{J}$, and using Faraday’s law, $\partial_t \mathbf{B} = -\nabla \times \mathbf{E}$, gives

$$\frac{\partial u}{\partial t} = \frac{1}{\mu_0} \Big(\mathbf{E} \cdot (\nabla \times \mathbf{B}) - \mathbf{B} \cdot (\nabla \times \mathbf{E}) \Big) - \mathbf{J} \cdot \mathbf{E},$$

and the same identity used above, now with the roles of $\mathbf{E}$ and $\mathbf{B}$ swapped, turns the first piece into a divergence:

$$\frac{\partial u}{\partial t} = -\mathbf{J} \cdot \mathbf{E} - \nabla \cdot \left(\frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right).$$

6 Poynting theorem

Define the Poynting vector,

$$\boxed{\mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0}.}$$

The local conservation law is

$$\boxed{\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E}.}$$

Integrating over a volume,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V u \, \mathrm{d}^3x + \oint_{\partial V} \mathbf{S} \cdot \mathrm{d}\mathbf{a} = -\frac{\mathrm{d}W}{\mathrm{d}t}.$$

Thus the change in electromagnetic energy inside a volume equals the energy flux through its boundary plus the work performed on charged matter.

7 Covariant expressions

Using the field components fixed in the conventions,

$$F^{0i} = \frac{E^i}{c}, \qquad F^{ij} = \epsilon^{ijk} B_k,$$

summing over all pairs $\mu\nu$ counts each antisymmetric pair $(0i)$ and $(i0)$ alike, giving a factor of 2 for the electric part, and similarly for the magnetic part,

$$F^{\mu\nu} F_{\mu\nu} = 2 \left(-\frac{|\mathbf{E}|^2}{c^2} + |\mathbf{B}|^2 \right).$$

Moreover, since $F^{00} = 0$, only the spatial index contributes to

$$F^{0\alpha} F^{0}{}_{\alpha} = \frac{|\mathbf{E}|^2}{c^2}.$$

The energy-momentum tensor is

$$T^{\alpha\beta} = \frac{1}{\mu_0} \left(F^{\alpha\gamma} F^{\beta}{}_{\gamma} - \frac{1}{4} \eta^{\alpha\beta} F^{\mu\nu} F_{\mu\nu} \right).$$

For $\alpha = \beta = 0$, using $\eta^{00} = -1$,

$$T^{00} = \frac{1}{\mu_0} \left(F^{0\alpha} F^{0}{}_{\alpha} + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right) = \frac{1}{\mu_0} \left(\frac{|\mathbf{E}|^2}{c^2} + \frac{1}{2} \left(-\frac{|\mathbf{E}|^2}{c^2} + |\mathbf{B}|^2 \right) \right) = \frac{1}{2\mu_0} \left(\frac{|\mathbf{E}|^2}{c^2} + |\mathbf{B}|^2 \right) = u.$$

For $\alpha = 0, \beta = i$, the second term drops out since $\eta^{0i} = 0$, and only $F^{00} = 0$ fails to contribute to the sum over γ, leaving

$$T^{0i} = \frac{1}{\mu_0} F^{0j} F^{i}{}_{j} = \frac{1}{\mu_0} \frac{E^j}{c} \epsilon^{ijk} B_k = \frac{1}{\mu_0 c} (\mathbf{E} \times \mathbf{B})^i = \frac{S^i}{c}.$$

Thus $T^{00} = u$ and $T^{0i} = S^i/c$ are, respectively, the time-time and time-space components of a single covariant object.

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