Calculus on Minkowski Spacetime

Sandro Vitenti

Scope

These notes continue the geometry module and follow the third lecture in a more complete way. The focus is to connect:

- curves, fields, and observer kinematics,
- tensor/form language in Minkowski spacetime,
- integral and differential viewpoints,
- the geometric structure behind Maxwell theory.

The main message is that much of electrodynamics is geometry plus source content.

Why Use a Covariant View

Coordinate systems are useful, but they can hide structure. A covariant formulation keeps only geometric information that is physically meaningful.

In practice:

- coordinates are tools, not physics;
- observer choices change component values but not geometric objects;
- projection operators make observer dependence explicit and controlled.

This is especially important when moving between inertial observers or when comparing electric/magnetic splits.

Curves, Fields, and Their Relation

Keep the distinction clear:

- a **curve** is $x^\mu(\lambda)$,
- a **tangent** to that curve is $\mathrm{d}x^\mu/\mathrm{d}\lambda$,
- a **vector field** is $V^\mu(x)$ defined at all spacetime points.

Integral curves of a field are determined by

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = V^\mu(x(\lambda)).$$

So a field generates a congruence of curves, while a single curve gives one worldline.

> **Note**
>
> Using the same symbols in both contexts is common in physics, but conceptually they are not the same object.

Proper Time and Affine Parametrization

For timelike worldlines, proper time τ satisfies

$$g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = -1.$$

If $u^\mu := \mathrm{d}x^\mu/\mathrm{d}\tau$, then

$$g_{\mu\nu}u^\mu u^\nu = -1.$$

This is the normalization associated with affine/proper-time parametrization. For non-affine parameters, this normalization is not constant in general.

In many calculations one can still parametrize by coordinate time, but proper time keeps covariance and normalization transparent.

Observer Field and Spatial Projector

Let u^μ be a unit timelike observer field. Define

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu.$$

Then

$$h_{\mu\nu}u^\nu = 0, \qquad h^\mu{}_\alpha h^\alpha{}_\nu = h^\mu{}_\nu.$$

So $h^\mu{}_\nu$ projects onto the local spatial subspace orthogonal to u.

For any vector A^μ,

$$A^\mu = -(u\cdot A)u^\mu + \bar{A}^\mu, \qquad \bar{A}^\mu := h^\mu{}_\nu A^\nu,$$

which separates temporal and spatial parts relative to that observer.

Rapidity and Relative Motion

For timelike unit vectors u^μ and v^μ,

$$-(u\cdot v) = \cosh\xi, \qquad \sqrt{h_{\mu\nu}v^\mu v^\nu} = \sinh\xi,$$

where ξ is rapidity.

In standard speed language:

$$anh\xi = \frac{v}{c}, \qquad \cosh\xi = \gamma.$$

This is the coordinate-free content of Lorentz boosts. The pair $\{\sinh\xi, \cosh\xi\}$ replaces Euclidean trigonometric decomposition by a hyperbolic one adapted to Minkowski signature.

Electromagnetic Split Relative to an Observer

Given Faraday tensor $F_{\mu\nu} = -F_{\nu\mu}$, define

$$E_\mu := F_{\mu\nu}u^\nu, \qquad E_\mu u^\mu = 0,$$

and the spatial magnetic 2-form

$$B_{\alpha\beta} := h_\alpha{}^\mu h_\beta{}^\nu F_{\mu\nu}.$$

Then

$$F_{\alpha\beta} = B_{\alpha\beta} + u_\alpha E_\beta - u_\beta E_\alpha.$$

So electric and magnetic fields are observer-dependent projections of one spacetime tensor, not two independent fundamental fields.

Differential Forms: Geometric Degrees

Useful interpretation by degree:

- 0-form: scalar field,
- 1-form: linear functional on vectors,
- 2-form: oriented area density,
- 3-form: oriented volume density,
- 4-form (in spacetime): oriented spacetime-volume density.

Wedge products encode oriented geometric content. Antisymmetry means repeated directions collapse to zero, which is exactly what should happen for oriented areas/volumes.

Exterior Derivative

For a scalar f,

$$\mathrm{d}f = (\partial_\mu f)\,\mathrm{d}x^\mu.$$

For a 1-form $\omega = \omega_\mu \mathrm{d}x^\mu$,

$$\mathrm{d}\omega = (\partial_\nu \omega_\mu)\,\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu.$$

For a 2-form $\beta = \frac{1}{2}\beta_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$,

$$\mathrm{d}\beta = \frac{1}{2}(\partial_\alpha \beta_{\mu\nu})\,\mathrm{d}x^\alpha \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu.$$

Key identity:

$$\mathrm{d}^2 = 0,$$

for smooth fields (commuting mixed partial derivatives).

Integrals of Forms and Parametrization Independence

A form is integrated on a geometric domain of matching dimension:

- 1-form on curves,
- 2-form on surfaces,
- 3-form on volumes.

The parametrization supplies tangent vectors and Jacobian factors. Under reparametrization, these compensate exactly, so the geometric integral is unchanged.

This is why one writes $\int_\Sigma \omega$ for a domain Σ itself, rather than tying meaning to one specific coordinate chart.

Stokes Theorem as the Unifying Principle

Prototype in 1D:

$$\int_a^b \mathrm{d}f = f(b) - f(a).$$

General form:

$$\int_{\partial\Omega} \omega = \int_\Omega \mathrm{d}\omega.$$

Classical theorems appear as special cases:

- gradient/fundamental theorem (1D),
- Green/Stokes (boundary curve vs enclosed surface),
- divergence theorem (closed surface vs enclosed volume).

So the many vector-calculus integral identities are one geometric identity written at different form degrees.

Worked Integration Examples

This section follows the lecture emphasis: concrete integral identities before moving to full Maxwell equations.

Example 1: Fundamental Theorem as 1D Stokes

Let $f(x) = x^3 - 2x$. Then

$$\mathrm{d}f = (3x^2 - 2)\,\mathrm{d}x.$$

On $[0, 2]$,

$$\int_0^2 \mathrm{d}f = \int_0^2 (3x^2 - 2)\,\mathrm{d}x = \left[x^3 - 2x\right]_0^2 = 4.$$

Boundary evaluation gives the same result:

$$f(2) - f(0) = 4 - 0 = 4.$$

Example 2: Green's Theorem from a 1-Form

In $\mathbb{R}^2$, take

$$\omega = P\,\mathrm{d}x + Q\,\mathrm{d}y \quad \text{with} \quad P = -y,\ Q = x.$$

Then

$$\mathrm{d}\omega = (\partial_x Q - \partial_y P)\,\mathrm{d}x \wedge \mathrm{d}y = (1 - (-1))\,\mathrm{d}x \wedge \mathrm{d}y = 2\,\mathrm{d}x \wedge \mathrm{d}y.$$

Let Ω be the unit disk. Stokes gives

$$\oint_{\partial\Omega} \omega = \iint_{\Omega} \mathrm{d}\omega.$$

Right side:

$$\iint_{\Omega} 2\,\mathrm{d}A = 2\pi.$$

Left side, with $x = \cos t$, $y = \sin t$:

$$\omega = -y\,\mathrm{d}x + x\,\mathrm{d}y = \sin^2 t\,\mathrm{d}t + \cos^2 t\,\mathrm{d}t = \mathrm{d}t,$$

so

$$\oint_{\partial\Omega} \omega = \int_0^{2\pi} \mathrm{d}t = 2\pi.$$

Example 3: 3D Stokes Theorem (Circulation-Curl)

Let

$$\mathbf{A} = (-y, x, 0),$$

with associated 1-form

$$\omega = -y\,\mathrm{d}x + x\,\mathrm{d}y.$$

Then

$$\mathrm{d}\omega = 2\,\mathrm{d}x \wedge \mathrm{d}y,$$

which corresponds to $\nabla \times \mathbf{A} = (0, 0, 2)$.

Choose Σ as the unit disk in the xy-plane, oriented upward. Then

$$\int_{\partial\Sigma} \omega = \int_{\Sigma} \mathrm{d}\omega = 2\pi.$$

This is exactly the circulation-curl theorem in form language.

Example 4: Divergence Theorem from a 2-Form

Let

$$\mathbf{B} = (x, y, z).$$

Associate the 2-form

$$\beta = B_x\,\mathrm{d}y \wedge \mathrm{d}z + B_y\,\mathrm{d}z \wedge \mathrm{d}x + B_z\,\mathrm{d}x \wedge \mathrm{d}y.$$

Then

$$\mathrm{d}\beta = (\nabla \cdot \mathbf{B})\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

Since $\nabla \cdot \mathbf{B} = 1 + 1 + 1 = 3$,

$$\iiint_V \mathrm{d}\beta = 3\,\mathrm{Vol}(V).$$

For the unit ball V, $\mathrm{Vol}(V) = \frac{4\pi}{3}$, so

$$\iiint_V \mathrm{d}\beta = 4\pi.$$

Hence, by Stokes in degree 2,

$$\iint_{\partial V} \beta = 4\pi,$$

which matches the classical outward flux of $\mathbf{B}$ through the unit sphere.

Example 5: Reparametrization Invariance (Curve Integral)

Take the 1-form

$$\omega = x\,\mathrm{d}x,$$

and the curve from $x = 0$ to $x = 1$.

Using parameter $t \in [0, 1]$, $x = t$:

$$\int_\gamma \omega = \int_0^1 t\,\mathrm{d}t = \frac{1}{2}.$$

Using parameter $s \in [0, 1]$, $x = s^3$:

$$\int_\gamma \omega = \int_0^1 s^3\,(3s^2)\,\mathrm{d}s = \int_0^1 3s^5\,\mathrm{d}s = \frac{1}{2}.$$

Same geometric curve, same oriented endpoints, same integral.

Divergence vs Curl in Form Language

In 3D Euclidean slices:

- from a 1-form, d gives a 2-form (curl-type content),
- from a 2-form, d gives a 3-form (divergence-type content).

The same operator d does both jobs, depending on degree. This is a major reason forms simplify electrodynamics.

Electric vs Magnetic Geometric Nature

Geometrically:

- electric field is naturally a spatial 1-form,
- magnetic field is naturally a spatial 2-form.

The usual magnetic pseudovector is a 3D dual representation. It is convenient, but it hides that magnetism is fundamentally area-oriented in this formalism.

Practical Takeaways

- Covariance helps separate geometric invariants from coordinate artifacts.
- Observer projection is the right way to discuss frame-dependent fields.
- Exterior derivative plus Stokes gives a unified differential/integral picture.
- The integral theorems are best seen as one geometric statement in different form degrees.

Exercises

1. Let u^μ be unit timelike and $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$. Prove $h^\mu{}_\nu$ is a projector and $h_{\mu\nu}u^\nu = 0$.
2. For any vector A^μ, verify
$$A^\mu = -(u \cdot A)u^\mu + h^\mu{}_\nu A^\nu.$$
3. For timelike unit vectors u^μ, v^μ, derive the rapidity relations from the decomposition of v^μ into temporal/spatial parts relative to u^μ.
4. Show from antisymmetry that $E_\mu u^\mu = 0$ when $E_\mu = F_{\mu\nu}u^\nu$.
5. Starting from the projector definition of $B_{\alpha\beta}$, prove
$$F_{\alpha\beta} = B_{\alpha\beta} + u_\alpha E_\beta - u_\beta E_\alpha.$$
6. Compute $\mathrm{d}\omega$ for a generic 1-form and verify explicitly in flat coordinates that $\mathrm{d}^2\omega = 0$.
7. Repeat Example 2 using the square $[-1, 1] \times [-1, 1]$ instead of the unit disk and confirm both sides agree.
8. For $\mathbf{A} = (-y, x, 0)$ and a disk of radius R, compute $\int_{\partial\Sigma} \omega$ and $\int_\Sigma \mathrm{d}\omega$ explicitly and show both equal $2\pi R^2$.
9. For $\mathbf{B} = (x, y, z)$ and a ball of radius R, verify
$$\iint_{\partial V} \beta = 4\pi R^3.$$
10. Explain, with one explicit example, how the same identity
$$\int_{\partial\Omega} \omega = \int_\Omega \mathrm{d}\omega$$
reproduces a familiar vector-calculus theorem.
11. Discuss why representing $\mathbf{B}$ as a pseudovector is special to 3 spatial dimensions.