

Accelerated Coordinates and Radiation from Moving Charges

Sandro Vitenti

Scope

These notes conclude the discussion of accelerated observers in flat spacetime and begin the study of radiation from moving charges.

The goals are:

- understand why Rindler coordinates are special;
- distinguish accelerated congruences from Born-rigid congruences;
- construct accelerated coordinates with identical proper acceleration;
- introduce the electromagnetic energy-momentum tensor;
- identify the radiative part of the Liénard–Wiechert field;
- derive the physical meaning of energy and momentum transport by the field.

Rindler Coordinates Revisited

Consider Minkowski spacetime with coordinates (T, X, Y, Z) and metric

$$ds^2 = -dT^2 + dX^2 + dY^2 + dZ^2.$$

Define

$$T = \rho \sinh(a\eta), \quad X = \rho \cosh(a\eta).$$

Substituting,

$$ds^2 = -(a\rho)^2 d\eta^2 + d\rho^2 + dY^2 + dZ^2.$$

Worldlines $\rho = \text{const}$ satisfy

$$X^2 - T^2 = \rho^2.$$

These are hyperbolae.

Using proper time,

$$d\tau = a\rho d\eta,$$

the four-velocity is

$$u^\mu = \left(\cosh \frac{\tau}{\rho}, \sinh \frac{\tau}{\rho} \right),$$

and the four-acceleration is

$$a^\mu = \frac{1}{\rho} \left(\sinh \frac{\tau}{\rho}, \cosh \frac{\tau}{\rho} \right).$$

Hence

$$a^\mu a_\mu = \frac{1}{\rho^2}.$$

The proper acceleration is

$$\alpha = \frac{1}{\rho}.$$

i Note

Different Rindler observers have different proper accelerations. The observer closest to the horizon accelerates the most.

Born Rigidity

For a congruence with four-velocity u^μ , define

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu.$$

The spatial deformation tensor is

$$\Theta_{\mu\nu} = h_\mu^\alpha h_\nu^\beta \nabla_{(\alpha} u_{\beta)}.$$

Born rigidity requires

$$\Theta_{\mu\nu} = 0.$$

This means that neighboring observers maintain constant proper separation in their instantaneous rest frame. Rindler observers satisfy this condition.

Why Equal Acceleration is Incompatible with Born Rigidity

Suppose every observer had the same proper acceleration,

$$a^\mu a_\mu = a_0^2.$$

Imagine two neighboring observers separated by a fixed proper distance.

After a short interval, the rear observer must have acquired a slightly larger velocity than the front observer if the separation is to remain fixed in the instantaneous rest frame.

Therefore the proper acceleration must vary along the congruence.

The result is that translational Born-rigid acceleration necessarily requires an acceleration gradient.

Rindler coordinates realize exactly this behavior.

Accelerated Coordinates with Equal Proper Acceleration

Start from the uniformly accelerated trajectory

$$T_0(\tau) = \frac{1}{a} \sinh(a\tau), \quad X_0(\tau) = \frac{1}{a} \cosh(a\tau).$$

Translate the trajectory by a constant spatial label ξ :

$$T(\tau, \xi) = \frac{1}{a} \sinh(a\tau),$$

$$X(\tau, \xi) = \xi + \frac{1}{a} \cosh(a\tau).$$

Every worldline has

$$a^\mu a_\mu = a^2.$$

The different observers therefore possess identical proper acceleration.

Computing

$$dT = \cosh(a\tau)d\tau,$$

$$dX = d\xi + \sinh(a\tau)d\tau,$$

gives

$$ds^2 = -d\tau^2 + 2 \sinh(a\tau)d\tau d\xi + d\xi^2.$$

The metric has

$$g_{0i} \neq 0.$$

The congruence is accelerated but not of the Rindler type.

Comparison with Rindler

Property	Rindler	Equal-Acceleration Family
Proper acceleration	varies as $1/\rho$	constant
Born rigid	yes	no
Horizon	yes	no
g_{0i}	0	nonzero
Stationary	yes	no

Transition to Electrodynamics

Accelerated observers reveal nontrivial geometric effects.

Accelerated charges reveal something more dramatic:

they radiate.

The key question is how to identify energy and momentum carried by the electromagnetic field.

Electromagnetic Energy-Momentum Tensor

The electromagnetic field tensor is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

We work in SI units with $c = 1$, so that $\epsilon_0\mu_0 = 1$ and $\epsilon_0 = 1/\mu_0$; the source equation then reads $\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu$. The energy-momentum tensor is

$$T^{\mu\nu} = \frac{1}{\mu_0} \left(F^{\mu\alpha} F^\nu{}_\alpha - \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right).$$

This object contains:

- energy density;
- momentum density;
- stresses;
- energy flux.

All local conservation laws are encoded in a single tensor equation.

Physical Interpretation of the Components

Energy Density

For an inertial observer,

$$T^{00} = \frac{1}{2\mu_0} (E^2 + B^2).$$

This is the electromagnetic energy density.

Momentum Density

The spatial momentum density is

$$T^{i0} = \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B})^i.$$

Energy flux and momentum density coincide because electromagnetic radiation travels at the speed of light.

Poynting Vector

The energy flux is

$$S^i = T^{0i}.$$

Therefore

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}.$$

Maxwell Stress Tensor

The spatial components are

$$T^{ij} = \frac{1}{\mu_0} \left(-E^i E^j - B^i B^j + \frac{1}{2} (E^2 + B^2) \delta^{ij} \right).$$

These describe pressure and momentum transport.

Poynting's Theorem

We now derive the local statement of energy conservation for the electromagnetic field directly from Maxwell's equations. In the SI conventions fixed above ($c = 1$, $\epsilon_0 = 1/\mu_0$), the equations containing the sources are

$$\nabla \cdot \mathbf{E} = \mu_0 \rho, \quad \nabla \times \mathbf{B} - \partial_t \mathbf{E} = \mu_0 \mathbf{J},$$

and the homogeneous equations are

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0.$$

The rate at which the field does work on the charges, per unit volume, is $\mathbf{J} \cdot \mathbf{E}$. Using Ampère's law to eliminate \mathbf{J} ,

$$\mathbf{J} \cdot \mathbf{E} = \frac{1}{\mu_0} [\mathbf{E} \cdot (\nabla \times \mathbf{B}) - \mathbf{E} \cdot \partial_t \mathbf{E}].$$

The vector identity

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B}),$$

combined with Faraday's law $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$, gives

$$\mathbf{E} \cdot (\nabla \times \mathbf{B}) = -\mathbf{B} \cdot \partial_t \mathbf{B} - \nabla \cdot (\mathbf{E} \times \mathbf{B}).$$

Substituting and collecting the time derivatives,

$$\mathbf{J} \cdot \mathbf{E} = -\partial_t \left[\frac{1}{2\mu_0} (E^2 + B^2) \right] - \nabla \cdot \left(\frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \right).$$

Introducing the field energy density and the Poynting vector,

$$u = \frac{1}{2\mu_0} (E^2 + B^2), \quad \mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B},$$

this rearranges into **Poynting's theorem**:

$$\partial_t u + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E}.$$

Integral Form and Interpretation

Integrating over a fixed volume V and applying the divergence theorem,

$$\frac{d}{dt} \int_V u dV + \oint_{\partial V} \mathbf{S} \cdot d\mathbf{A} = - \int_V \mathbf{J} \cdot \mathbf{E} dV.$$

Each term has a direct physical meaning:

- the first term is the rate of change of the electromagnetic energy stored in V ;
- the surface term is the energy flux leaving V through its boundary, carried by the field;
- the term $\int_V \mathbf{J} \cdot \mathbf{E} dV$ is the rate of work done by the field on the charges, i.e. the energy transferred from field to matter.

The theorem states that the field energy in V can change only by flowing across the boundary or by being delivered to the charges. This identifies \mathbf{S} as the energy flux density of the electromagnetic field.

Covariant Form of Poynting's Theorem

Poynting's theorem and the analogous statement of momentum conservation are unified in a single covariant equation. Starting from Maxwell's equations in covariant form,

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu,$$

and using the Bianchi identity $\partial_{[\lambda} F_{\mu\nu]} = 0$ to dispose of the purely field-dependent terms, the divergence of the energy-momentum tensor is

$$\partial_\mu T^{\mu\nu} = -F^{\nu\lambda} J_\lambda.$$

The right-hand side is (minus) the **Lorentz four-force density**

$$f^\nu = F^{\nu\lambda} J_\lambda,$$

the rate at which four-momentum is transferred from the field to the charges. The single equation $\partial_\mu T^{\mu\nu} = -f^\nu$ therefore contains both energy and momentum balance, recovered by fixing the free index ν .

Energy Component ($\nu = 0$)

With $T^{00} = u$ and $T^{i0} = S^i$, the left-hand side is

$$\partial_\mu T^{\mu 0} = \partial_t u + \nabla \cdot \mathbf{S},$$

while the force density gives $f^0 = F^{0\lambda} J_\lambda = \mathbf{E} \cdot \mathbf{J}$. Hence

$$\partial_t u + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E},$$

which is exactly Poynting's theorem. The time component of $\partial_\mu T^{\mu\nu} = -f^\nu$ is the statement of energy conservation.

Momentum Components ($\nu = i$)

With $T^{0i} = S^i$ playing the role of the momentum density, the spatial components read

$$\partial_t S^i + \partial_j T^{ji} = -f^i, \quad f^i = (\rho \mathbf{E} + \mathbf{J} \times \mathbf{B})^i,$$

where the source is the ordinary Lorentz force density acting on the charges. Introducing the Maxwell stress tensor $\sigma^{ij} = -T^{ij} = \frac{1}{\mu_0} [E^i E^j + B^i B^j - \frac{1}{2} \delta^{ij} (E^2 + B^2)]$, this becomes

$$\partial_t S^i = \partial_j \sigma^{ij} - (\rho \mathbf{E} + \mathbf{J} \times \mathbf{B})^i,$$

the conservation of momentum: the field momentum density \mathbf{S} changes through the momentum flux carried by the stresses σ^{ij} and through the momentum delivered to matter by the Lorentz force.

Vacuum

In the absence of sources,

$$J^\mu = 0 \quad \implies \quad \partial_\mu T^{\mu\nu} = 0,$$

so electromagnetic energy and momentum are separately conserved. This is the situation relevant far from a radiating source, where the field propagates freely and $T^{\mu\nu}$ describes energy and momentum streaming outward to infinity.

Liénard–Wiechert Fields

For a moving charge,

$$R^\mu = x^\mu - z^\mu(\tau_r),$$

where τ_r is the retarded proper time.

The field naturally separates into

$$F_{\mu\nu} = F_{\mu\nu}^{\text{vel}} + F_{\mu\nu}^{\text{acc}}.$$

The scaling with distance is

$$F^{\text{vel}} \sim \frac{1}{R^2},$$

$$F^{\text{acc}} \sim \frac{1}{R}.$$

The radiative contribution is therefore the acceleration field.

Radiation Zone

Far from the source,

$$\mathbf{B} = \hat{\mathbf{n}} \times \mathbf{E},$$

and

$$|\mathbf{E}| = |\mathbf{B}|.$$

Then

$$u = T^{00} = \frac{E^2}{\mu_0},$$

and

$$\mathbf{S} = \frac{E^2}{\mu_0} \hat{\mathbf{n}}.$$

Energy flows radially outward.

Covariant Radiation Tensor

The radiation contribution to the energy-momentum tensor has the form

$$T_{\text{rad}}^{\mu\nu} \propto \frac{a^2(R \cdot u)^2 - (R \cdot a)^2}{(R \cdot u)^6} R^\mu R^\nu.$$

Important observations:

- proportional to $R^\mu R^\nu$;
- null;
- directed along outgoing light rays;
- survives at arbitrarily large distances.

This is the field responsible for energy loss by the source.

Relativistic Larmor Formula

Integrating the radiative flux over a large sphere yields

$$P = \frac{\mu_0 q^2}{6\pi} a^\mu a_\mu.$$

In the instantaneous rest frame,

$$P = \frac{\mu_0 q^2}{6\pi} a^2.$$

This is the relativistic form of Larmor's formula.

Summary

The Rindler congruence is a special Born-rigid accelerated frame whose proper acceleration varies spatially.

Uniform proper acceleration for all observers can be achieved, but only at the cost of losing Born rigidity.

For electrodynamics, the energy-momentum tensor provides the covariant description of energy density, momentum density, stresses, and fluxes.

The radiative part of the Liénard–Wiechert field scales as $1/R$ and produces a finite energy flux at infinity, leading to the Larmor radiation formula.